Math 175 Spring 2007: Exam 2 Part I
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03/23/07

Name: Solution

Part I: You can use a calculator

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 5x_2 + 6x_3 \end{bmatrix}$.

   (a) Determine whether T is one-to-one or not. To receive complete credit, you must show all your work.

   (b) Determine whether T is onto or not. To receive complete credit, you must show all your work.

The matrix representation of T is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. So, Rank $A = 2$.

Since the domain of T is $\mathbb{R}^3$ and Rank $A = 2$ $(\neq 3)$, we can conclude that

T is not 1-1.

Since rank of $A = 2$ and the range of T is $\mathbb{R}^2$, we can conclude that

T is onto.

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Part II: You can not use a calculator
Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (10 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Find AB and BA if they exist.

2. (10 points) Prove that if A is an invertible matrix such that $AB = AC$, then $B = C$.

3. (10 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Use the definition of linear transformation to prove that $T + U$ is a linear transformation. Recall that $(T+U)(\vec{v}) = T(\vec{v}) + U(\vec{v})$.

4. (10 points) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

   (a) Determine whether A is invertible or not. If A is invertible, please find A^{-1}.

   (b) Find $det(A)$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a function defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_1x_2 \\ 4x_3 \end{bmatrix}$.
   Prove or disprove: T is a linear transformation.

1) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+3 & 3 \\ 4+6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 6 \end{bmatrix}$.

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1+4 & 2+5 & 3+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 6 & 9 \end{bmatrix}$.

2) Proof Let A be an invertible matric such that $AB = AC$.

Hence, $A^{-1}(AB) = A^{-1}(AC)$

$$(A^{-1}A)B = (A^{-1}A)C$$

$$IB = IC$$

$$B = C.$$

3). Proof 1) We will show that $(T+U)(\vec{a}+\vec{b}) = (T+U)(\vec{a}) + (T+U)(\vec{b})$ for all $\vec{a}, \vec{b} \in \mathbb{R}^n$.

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. $(T+U)(\vec{a}+\vec{b}) = T(\vec{a}+\vec{b}) + U(\vec{a}+\vec{b})$

$= T(\vec{a}) + T(\vec{b}) + U(\vec{a}) + U(\vec{b})$ (since T and U are lin. transf)

$= T(\vec{a}) + U(\vec{a}) + T(\vec{b}) + U(\vec{b})$

$= (T+U)(\vec{a}) + (T+U)(\vec{b})$.

2) We will show that $(T+U)(\alpha\vec{a}) = \alpha(T+U)(\vec{a})$ for all $\vec{a} \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

Let $\vec{a} \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

$(T+U)(\alpha\vec{a}) = T(\alpha\vec{a}) + U(\alpha\vec{a}) = \alpha T(\vec{a}) + \alpha U(\vec{a})$ since T and U are lin. transf.

$= \alpha(T(\vec{a}) + U(\vec{a}))$

$= \alpha(T+U)(\vec{a})$.

By 1) & 2), we can conclude that $T+U$ is a lin. transf.

4). $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \frac{1}{3}R_2 \\ -R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$

$R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & 1 \\ 0 & 1 & 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$

Hence A is invertible and $A^{-1} = \begin{bmatrix} 2/3 & 1/3 & 1 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$\det A = \det \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = (-1)(2+1) = -3.$

5). Since $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

and $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 + 3(1)(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$,

we can conclude that T does not preserve addition and

T is not a linear transformation.