Math 175 Spring 2007: Exam 1 Part I
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02/23/07

Name: Solution

Part I: You can use a calculator

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (10 points) Determine if $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$. To receive complete credit, you must show all your work.

2. (10 points) Determine whether the vector $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. To receive complete credit, you must show all your work.

1) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$. Since Rank $A = 3$, we can conclude that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

2) Since $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 5 & 3 & -1 & 4 \\ 2 & 4 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, we can conclude that

$\downarrow$

$0 = 1$

$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Math 175 Spring 2007: Exam 1 Part II

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Name: Solution

Part II: You can not use a calculator

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (10 points) Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be a linearly independent subset of $\mathbb{R}^n$. Prove that $\{\vec{u}, \vec{u}-\vec{v}, \vec{w}\}$ is linearly independent.

2. (10 points) Let $\vec{u}$, $\vec{v}$ be any vectors in $\mathbb{R}^n$. Prove that

$$Span\{\vec{u}, \vec{v}\} = Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\}.$$

3. (10 points) Use Gaussian Elimination to determine whether the given system is consistent or not. If the system is consistent, find its general solution.

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= -2 \\ 2x_2 - 3x_3 - 12x_4 &= -3 \\ x_1 - x_3 + 6x_4 &= 0 \end{aligned}$$

1) Proof Assume that $\{\vec{u}, \vec{v}, \vec{w}\}$ is a lin. indep subset of $\mathbb{R}^n$.

Set $a\vec{u} + b(\vec{u}-\vec{v}) + c\vec{w} = \vec{0}$.

Hence, we have $(a+b)\vec{u} - b\vec{v} + c\vec{w} = 0$. Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin indep, we can conclude that $a+b=0$, $-b=0$, $c=0$. This implies that $a=b=c=0$, and $\{\vec{u}, \vec{u}-\vec{v}, \vec{w}\}$ is linearly independent.

2) Proof First, we will show that $Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\} \subseteq Span\{\vec{u}, \vec{v}\}$.

Since $\vec{u}-2\vec{v}, \vec{u}+2\vec{v} \in Span\{\vec{u}, \vec{v}\}$, by Thm 1.6, we can conclude that $Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\} \subseteq Span\{\vec{u}, \vec{v}\}$. (*)

Next, we will show that $Span\{\vec{u}, \vec{v}\} \subseteq Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\}$.

Since $\vec{u} = \frac{1}{2}(\vec{u}-2\vec{v}) + \frac{1}{2}(\vec{u}+2\vec{v})$ and $\vec{v} = \frac{-1}{2}(\vec{u}-2\vec{v}) + \frac{1}{2}(\vec{u}+2\vec{v})$, we can conclude that $\vec{u}, \vec{v} \in Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\}$.

Furthermore, we have $Span\{\vec{u},\vec{v}\} \subseteq Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\}$. (**)

By (*) and (**), we can conclude that $Span\{\vec{u},\vec{v}\} = Span\{\vec{u}-2\vec{v}, \vec{u}+2\vec{v}\}$.

3) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & 2 & -3 & -12 & -3 \\ 1 & 0 & -1 & 6 & 0 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & 2 & -3 & -12 & -3 \\ 0 & -1 & 0 & 7 & 2 \end{array}\right]$

$R_2 \leftrightarrow R_3 \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & -1 & 0 & 7 & 2 \\ 0 & 2 & -3 & -12 & -3 \end{array}\right] \quad R_3 + 2R_2 \mapsto R_3 \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & -1 & 0 & 7 & 2 \\ 0 & 0 & -3 & 2 & 1 \end{array}\right]$

$\begin{array}{c} -\frac{1}{3} R_3 \\ -1R_2 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & -7 & -2 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right] \quad R_1 + R_3 \mapsto R_1 \left[\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{5}{3} & -\frac{7}{3} \\ 0 & 1 & 0 & -7 & -2 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right]$

$R_1 - R_2 \mapsto R_1 \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{16}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -7 & -2 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right]$ The system is consistent. Moreover,

Hence $x_1 = -\frac{16}{3} x_4 - \frac{1}{3}$, $x_2 = +\frac{7}{3} x_4 - 2$, $x_3 = \frac{2}{3} x_4 - \frac{1}{3}$.

Hence, the general solution is

$$x_1 = -\frac{16}{3} s - \frac{1}{3}, \quad x_2 = \frac{7}{3} s - 2, \quad x_3 = \frac{2}{3} s - \frac{1}{3} \text{ and } x_4 = s,$$

where s is any real number.