Solution: Section 4.2

2) Let $f(x) = e^{-x^2}$. Find a formula for a function F so that $F'(x) = f(x)$

<u>Proof</u> Set $F(x) = \int_0^x \underbrace{e^{-t^2}}_{=f(t)} dt$. Since $f(x) = e^{-x^2}$ is cont on $\mathbb{R}$,

by The Fundamental theorem of Calculus Part II, we have that
$F'(x) = e^{-x^2}$.

5 Suppose that f is continuously differentiable on $[a,b]$ and $f'(x)=0$ for all x. Prove that f is a constant fcn.

<u>Proof</u> Let $x, y \in [a,b]$. <u>Goal</u> We will show that $f(x) = f(y)$.

We may assume that $x < y$.

Since f is cont on $[x,y]$ and diff on $[x,y]$, by MVT, it follow that $\exists\, c \in [x,y]$ s.t

$$\frac{f(y) - f(x)}{y - x} = f'(c) = 0.$$

Hence $f(y) - f(x) = 0$. So $f(y) = f(x)$.

Since x, y are arbitrary elts of $[a,b]$, we can conclude that f is a constant fcn.

7) Let f and g be in $C^1(\mathbb{R})$, and suppose that $f(0)=g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Prove that $f(x) \le g(x)$ for all $x \ge 0$

Proof Since $g'(x) - f'(x) \ge 0$ for all $x \ge 0$ and $(g-f)' = g'-f'$ is continuous[redacted] diff on $\mathbb{R}$, by The Fundamental Theorem, Part I, we then have that for $x \ge 0$

$$\begin{aligned}(g-f)(x) - (g-f)(0) &= \int_0^x (g-f)'(t)\,dt \\ &= \int_0^x g'(t) - f'(t)\,dt \\ &\ge \int_0^x 0\,dt = 0.\end{aligned}$$

Hence $g(x) - f(x) - (g(0) - f(0)) \ge 0$.

Since $f(0) = g(0) = 0$, we can conclude that $g(x) \ge f(x)$

for all $x \ge 0$

12. Suppose that f and g are continuously diff on $[a,b]$. Prove that

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Proof Since f and g are diff on $[a,b]$, we have that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \text{ for all } x \in [a,b]$$

By The Fundamental Thm, part I, we have

$$\int_a^b (fg)'(x)dx = (fg)(b) - (fg)(a)$$

$$= f(b)g(b) - f(a)g(a). \quad (*)$$

Since $\int_a^b (fg)'(x)dx = \int_a^b f'(x)g(x) + f(x)g'(x)\, dx$

and $\underbrace{f'(x)g(x)}_{\text{cont on } [a,b]}$ and $\underbrace{f(x)g'(x)}_{\text{cont on } [a,b]}$ are Riemann integrable,

we can conclude that $\int_a^b (fg)'(x)dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)\,dx.$ $(**)$

By $(*)$, $(**)$, we can conclude that

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$