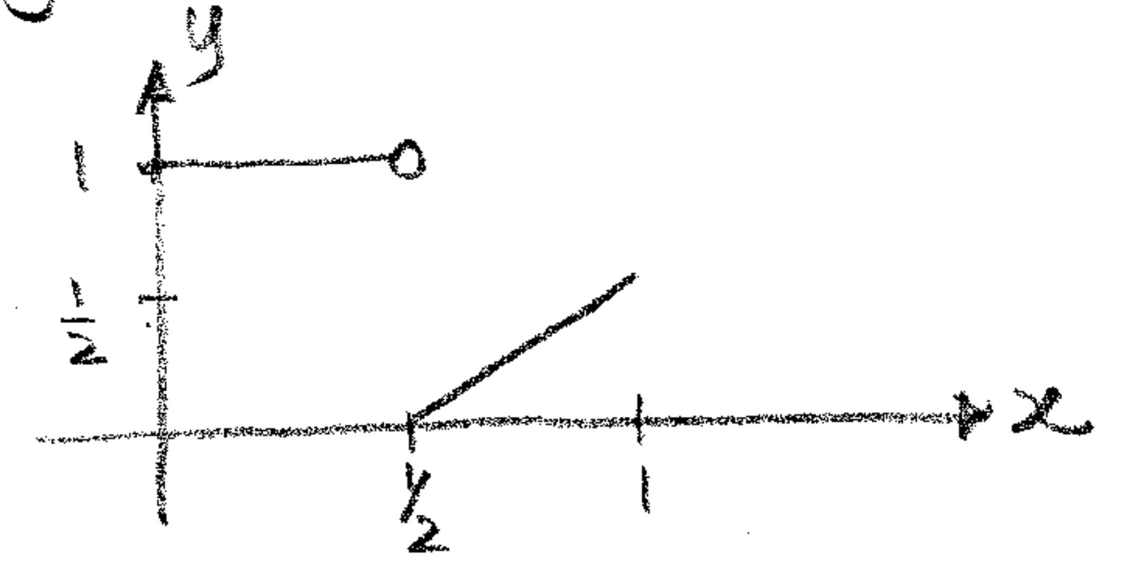
Solution for section 3.5

3) Let f be the function on $[0,1]$ given by

$$f(x)=\begin{cases} 1 & 0\le x<\frac{1}{2} \\ x-\frac{1}{2} & \frac{1}{2}\le x<1 \end{cases}$$



a) Prove that f is Riemann integrable without appealing to any theorems in this section.

Proof. Let $\varepsilon>0$. $\exists N\in\mathbb{N}$ s.t $\frac{3}{4N}<\varepsilon$.

Let P be a partition on $[0,1]$: $0=x_0<x_1<\dots<x_{2N}=1$ such that $x_i-x_{i-1}=\frac{1}{2N}$. Hence, $x_i=\frac{i}{2N}$. In particular $x_N=\frac{1}{2}$.

Since
$$U_p(f)=\sum_{i=1}^{2N}M_i(x_i-x_{i-1})=\sum_{i=1}^{N}\underbrace{M_i}_{1}(x_i-x_{i-1})+\sum_{i=N+1}^{2N}\underbrace{M_i}_{f(x_i)}(x_i-x_{i-1})$$
$$=\sum_{i=1}^{N}1\left(\frac{1}{2N}\right)+\sum_{i=N+1}^{2N}\left(\frac{i}{2N}-\frac{1}{2}\right)\frac{1}{2N}$$
$$=\frac{1}{2N}(N)+\frac{1}{(2N)^2}\left(\sum_{i=N+1}^{2N}i\right)-\frac{1}{2N}\left(\sum_{i=N+1}^{2N}\frac{1}{2}\right)$$
$$=\frac{1}{2}+\frac{1}{(2N)^2}\left(\sum_{i=N+1}^{2N}i\right)-\frac{1}{4N}(N)$$
$$=\frac{1}{4}+\frac{1}{(2N)^2}\sum_{i=N+1}^{2N}i,$$

and: $$L_p(f)=\sum_{i=1}^{2N}m_i(x_i-x_{i-1})=\sum_{i=1}^{N-1}\underbrace{m_i}_{1}(x_i-x_{i-1})+\underbrace{m_N}_{0}(x_N-x_{N-1})+\sum_{i=N+1}^{2N}\underbrace{m_i}_{f(x_{i-1})}(x_i-x_{i-1}).$$
$$=\sum_{i=1}^{N-1}\frac{1}{2N}+\sum_{i=N+1}^{2N}\left(\left(\frac{i-1}{2N}\right)-\frac{1}{2}\right)\frac{1}{2N}$$

$$= \frac{1}{2N}(N-1) + \frac{1}{(2N)^2}\sum_{i=N+1}^{2N}(i-1) - \frac{1}{4N}\sum_{i=N+1}^{2N} 1$$

$$= \frac{N-1}{2N} + \frac{1}{(2N)^2}\sum_{i=N+1}^{2N} i - \frac{1}{(2N)^2}N - \frac{N}{4N}$$

$$= \frac{1}{2} - \frac{1}{2N} + \frac{1}{(2N)^2}\sum_{i=N+1}^{2N} i - \frac{1}{4N} - \frac{1}{4}.$$

$$= \frac{1}{4} - \frac{3}{4N} + \frac{1}{(2N)^2}\sum_{i=1}^{2N} i ,$$

we have $$U_p(f) - L_p(f) = \frac{1}{4} + \frac{1}{(2N)^2}\sum_{i=N+1}^{2N} i - \left(\frac{1}{4} - \frac{3}{4N} + \frac{1}{(2N)^2}\sum_{i=1}^{2N} i\right)$$

$$= \frac{3}{4N} < \varepsilon .$$

Hence, f is Riemann integrable on $[0,1]$.

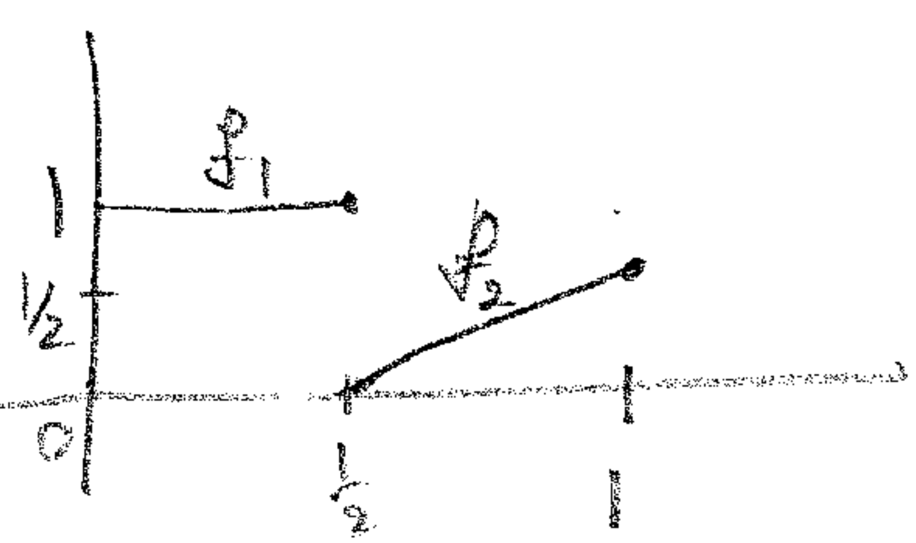
b) Which theorems in this section guarantee that f is Riemann integrable?

Answer Th^{m} 3.5.3

c) What is $\int_0^1 f(x)\,dx$?

By Cor 3.5.4, we have $\int_0^1 f(x)\,dx = \int_0^{1/2} f_1(x)\,dx + \int_{1/2}^1 f_2(x)\,dx$ where $f_1(x) = 1$ for all $x \in [0, \frac{1}{2}]$ and $f_2(x) = x - \frac{1}{2}$ for all $x \in [\frac{1}{2}, 1]$,

$\int_0^{1/2} f_1(x)\,dx$ is the area under the curve of $f_1(x)$ and x-axis between 0 and $\frac{1}{2}$.



Similarly $\int_{1/2}^1 f_2(x)\,dx$ is the area under the curve of $f_2(x)$ and x-axis between $\frac{1}{2}$ and 1.

Consequently, $\int_0^1 f(x)dx = \frac{1}{2}(1) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5/8$.

5) Suppose that f is defined in an open interval containing c and that (30) holds. Prove that f is continuous at c.

<u>Proof</u>. Let $\varepsilon > 0$. Since $\lim_{x \nearrow c} f(x) = f(c^-) = f(c)$, it follows that

$\exists \delta_1 > 0$ s.t for all x in Dom(f) and $x < c$

$$|x - c| \leq \delta_1 \implies |f(x) - f(c)| \leq \varepsilon.$$

Since $\lim_{x \searrow c} f(x) = f(c^+) = f(c)$, it follows that $\exists \delta_2 > 0$ s.t

for all x in Dom(f) and $x > c$

$$|x - c| < \delta_2 \implies |f(x) - f(c)| \leq \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $x \in$ Dom(f) s.t $|x - c| < \delta$

<u>case 1</u> $x < c$

Since $\delta \leq \delta_1$, it follows that $|x - c| \leq \delta_1$. Hence, $|f(x) - f(c)| \leq \varepsilon$.

<u>case 2</u> $x > c$

Since $\delta \leq \delta_2$, it follows that $|x - c| \leq \delta_2$. Therefore, $|f(x) - f(c)| \leq \varepsilon$.

Consequently if $|x - c| < \delta$ then $|f(x) - f(c)| \leq \varepsilon$. Furthermore,

$\lim_{x \to c} f(x) = f(c)$. So, f is continuous at c.

8) Show by example that a monotone increasing function on a finite interval can have infinitely many jump discontinuities.

<u>Example</u> Let $C = \{c_1, c_2, c_3, \ldots\}$ be a countable subset of (a, b)

Let $f(x) = \sum_{c_n < x} \frac{1}{2^n}$;

So,
$$f(x) = \begin{cases} 0 & \text{if } a \le x \le c_1 \\ \frac{1}{2} & \text{if } c_1 < x \le c_2 \\ \frac{1}{2} + \frac{1}{2^2} & \text{if } c_2 < x \le c_3 \\ \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} & \text{if } c_3 < x \le c_4 \\ \vdots & \\ 1 & \text{if } x = b. \end{cases}$$

f is a monotone increasing function and f have jump discontinuities at c_i for all $i \ge 1$.