Section 3.3 Problem

1) Let $f(x) = x^2$ and let P be a partition of interval $[1,2]$ into subintervals of length 0.2. Compute $U_P(f)$ and $L_P(f)$.

2) Let $f(x) = \frac{1}{x}$. Find a partition P of the interval $[1,3]$ so that $U_P(f) - L_P(f) \le 10^{-2}$.

3) Prove that $e-1 \le \int_0^1 \sqrt{1+x}\, e^x\, dx \le \sqrt{2}(e-1)$. (Assume that $\lim_{k\to\infty} k(1-e^{1/k}) = -1$

4) Let f be a continuous function on $[a,b]$ and define

$F(x) \equiv \int_a^x f(t)\,dt$. Prove that F is Lipschitz continuous on $[a,b]$

1) let $f(x) = x^2$ and let P be a partition of interval $[1, 2]$ into subintervals of length 0.2. Compute $U_p(f)$ and $L_p(f)$.

Sol^n Let $P = \{1, 1.2, 1.4, 1.6, 1.8, 2\}$.
$$\underset{x_0}{1},\ \underset{x_1}{1.2},\ \underset{x_2}{1.4},\ \underset{x_3}{1.6},\ \underset{x_4}{1.8},\ \underset{x_5}{2}$$

$$U_p(f) = \sum_{i=1}^{5} M_i(x_i - x_{i-1}) = \sum_{i=1}^{5} f(x_i)\, 0.2$$
(since $f(x) = x^2$ is an increasing fcn. and $x_i - x_{i-1} = 0.2$)
$$= \sum_{i=1}^{5} x_i^2\, 0.2$$
$$= 0.2\left((1.2)^2 + (1.4)^2 + (1.6)^2 + (1.8)^2 + 2^2\right).$$

$$L_p(f) = \sum_{i=1}^{5} m_i(x_i - x_{i-1}) = \sum_{i=1}^{5} f(x_{i-1})\, 0.2$$
$$= 0.2\left(1^2 + (1.2)^2 + (1.4)^2 + (1.6)^2 + (1.8)^2\right).$$

2) Let $f(x) = \frac{1}{x}$. Find a partition P of the interval $[1, 3]$ so that

$U_p(f) - L_p(f) \leq 10^{-2}$.

Sol^n Let $P = \{x_0, x_1, \ldots, x_N\}$ so that $x_i - x_{i-1} = \frac{2}{N}$, with $x_0 = 1$, $x_N = 3$.

$$U_p(f) - L_p(f) = \sum_{i=1}^{N} f(x_{i-1})(x_i - x_{i-1}) - \sum_{i=1}^{N} f(x_i)(x_i - x_{i-1})$$
since f is a decreasing fcn.

$$= \sum_{i=1}^{N} \frac{1}{x_{i-1}} \frac{2}{N} - \sum_{i=1}^{N} \frac{1}{x_i} \frac{2}{N}$$

$$= \frac{2}{N} \sum_{i=1}^{N} \frac{1}{x_{i-1}} - \frac{1}{x_i}$$

$$= \frac{2}{N} \left(\left(1 - \frac{1}{1+\frac{2}{N}}\right) + \left(\frac{1}{1+\frac{2}{N}} - \frac{1}{1+\frac{4}{N}}\right) + \dots + \left(\frac{1}{1+\frac{(N-1)2}{N}} - \frac{1}{1+2}\right) \right)$$

$$= \frac{2}{N}\left(1 - \frac{1}{3}\right).$$

Hence $U_P(f) - L_P(f) \leq 10^{-2}$ when $\frac{2}{N}\left(1-\frac{1}{3}\right) \leq 10^{-2}$

$$\frac{2}{N}\left(\frac{2}{3}\right) \leq \frac{1}{100}$$

$$\frac{4}{3}(100) \leq N$$

~~If $N \geq \frac{400}{3}$ then~~

So, ~~a Partition $P = \{x_0, x_1, \dots, x_N\}$ satisfies the property that $U_p(f) - L_p(f) \leq 10^{-2}$~~

If $N \geq \frac{400}{3}$ then a partition $P = \{\underset{1}{\overset{\shortparallel}{x_0}}, x_1, \dots, \underset{3}{\overset{\shortparallel}{x_N}}\}$ s.t $x_i = 1 + i \cdot \frac{2}{N}$ satisfies the property that $U_p(f) - L_p(f) \leq 10^{-2}$.

3 Prove that $e-1 \le \int_0^1 \sqrt{1+x}\, e^x\,dx \le \sqrt{2}(e-1)$. (Assume that $\lim_{k\to\infty} k(1-e^{1/k}) = -1$).

Proof Since $0 \le x \le 1$, we have $1 \le 1+x \le 2$. Moreover $1 \le \sqrt{1+x} \le \sqrt{2}$.

This implies that $e^x \le \sqrt{1+x}\, e^x \le \sqrt{2}\, e^x$ for all $x \in [0,1]$

Hence $\int_0^1 e^x dx \le \int_0^1 \sqrt{1+x}\, e^x dx \le \int_0^1 \sqrt{2}\, e^x$. $(*)$

We will show that $\int_0^1 e^x dx = e-1$.

Let $P_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1\}$ and $x_i^* = x_i$

Then $S_k = \sum_{i=1}^{k} f(x_i)(x_i - x_{i-1})$

$$= \sum_{i=1}^{k} e^{x_i} \frac{1}{k} = \frac{1}{k} \sum_{i=1}^{k} e^{i/k} = \frac{1}{k} \sum_{i=1}^{k} \left(e^{1/k}\right)^i$$

$$= \frac{1}{k} \frac{e^{1/k}(1-e)}{1-e^{1/k}}$$

As $k\to\infty$, we know that the maximum length of the subintervals of P_k goes to zero. Moreover, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{e^{1/k}(1-e)}{k(1-e^{1/k})}$

$$= (1-e) \frac{\lim_{k\to\infty} e^{1/k}}{\lim_{k\to\infty} k(1-e^{1/k})}$$

$$= (1-e) \frac{1}{(-1)} = e-1.$$

By Cor 3.3.2, we can conclude that $\int_0^1 e^x dx = e - 1$

Furthermore, by (*), we can conclude that $e - 1 \leq \int_0^1 \sqrt{1+x}\, e^x dx$

$$\leq \sqrt{2}(e-1).$$

4) Let f be a continuous function on the interval $[a,b]$ and define

$F(x) \equiv \int_a^x f(t)\,dt$. Prove that F is Lipschitz continuous on $[a,b]$

<u>Proof</u>

Let $x, c \in [a,b]$ s.t $x > c$.

$$\text{Since } |F(x) - F(c)| = \left| \int_a^x f(t)\,dt - \int_a^c f(t)\,dt \right|$$

$$= \left| \int_c^x f(t)\,dt \right|$$

$$\leq (x-c) \sup_{[c,x]} |f(x)|$$

$$= |x-c| \sup_{[c,x]} |f(x)| \qquad = M$$

we can conclude that F is Lipschitz cont. on $[a,b]$.