Solution for section 3.3

3) Suppose that f is Riemann integrable on $[a,b]$ and $f(x) \geq 0$ for all x.

a) Prove that $\int_a^b f(x)dx \geq 0$.

Proof Recall $\int_a^b f(x)dx = \sup_P \{L_P(f)\} = \inf_P \{U_P(f)\}$.

1) We will show that 0 is a lowerbound of $\{U_P(f) \mid P \text{ is any partition on } [a,b]\}$.

Let P be a partition on $[a,b]$: $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

Then $U_P(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ where $M_i \equiv \sup_x \{f(x) \mid x_{i-1} \leq x \leq x_i\}$.

Since $f(x) \geq 0$ for all $x \in [a,b]$, we can conclude that for $1 \leq i \leq n$

we have $M_i \geq f(x) \geq 0$ for all $x \in [x_{i-1}, x_i]$.

Since $x_i - x_{i-1} > 0$ and $M_i \geq 0$ for all i s.t $1 \leq i \leq n$,
we can conclude that $U_P(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq 0$.

Since P is an arbitrary partition on $[a,b]$, we can conclude that 0 is a lowerbound of $\{U_P(f) \mid P \text{ is any partition on } [a,b]\}$.

2) We will show that $\int_a^b f(x)dx \geq 0$.

Since $\int_a^b f(x)dx$ is $\inf_P \{U_P(f)\}$ and 0 is a lower bound of $\{U_P(f) \mid P \text{ is a partition on } [a,b]\}$ we can conclude that $\int_a^b f(x)dx \geq 0$.

b) Prove that if $\int_a^b f(x)dx = 0$ and f is continuous then $f(x)=0$ for all x in $[a,b]$.

<u>Proof</u> Since f is continuous on $[a,b]$, it follows that there exists $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$. We will show that $f(c)=0$.

Assume that $f(c) \neq 0$. Hence, $f(c) > 0$.

Since f is continuous at c, it implies that with $\varepsilon = \frac{f(c)}{2}$ there exists $\delta > 0$ s.t for all $x \in [a,b]$:

$|x-c| \leq \delta \Longrightarrow |f(x) - f(c)| \leq f(c)/2$,

Consequently if $|x-c| \leq \delta$, we then have $-\frac{f(c)}{2} \leq f(x) - f(c) \leq \frac{f(c)}{2}$

$$(*) \quad \frac{f(c)}{2} \leq f(x) \leq \frac{3}{2}f(c).$$

Since f is cont on $[c-\delta, c+\delta]$, $[a, c-\delta]$, $[c+\delta, b]$, and $f(x) \geq 0$ for all $x \in [a,b]$, by 3(a) we then have that

$$\int_{c-\delta}^{c+\delta} f(x)dx \geq 0, \quad \int_a^{c-\delta} f(x)dx \geq 0, \quad \int_{c+\delta}^{b} f(x)dx \geq 0.$$

Since
$$\int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx \geq \int_{c-\delta}^{c+\delta} f(x)dx$$
$$\geq \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}dx \quad (\text{by } *)$$
$$= \frac{f(c)}{2} 2\delta,$$

we can conclude that $\int_a^b f(x)dx > 0$. This is a contradiction.

Hence $f(c)=0$. Since $0 \le f(x) \le f(c)$ for all x in $[a,b]$,

it follows immediately that $f(x)=0$ for all x in $[a,b]$.

c) Find a counterexample which shows that the conclusion of part b may not hold if the hypothesis of continuity is removed.

Example.

Let $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \text{ and } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2}. \end{cases}$

Let $\varepsilon > 0$. By AP, there exists $N \in \mathbb{N}$ s.t $\frac{1}{N} < \varepsilon$.

Let P be a partition on $[0,1]$: $0 = x_0 < x_1 < x_2 < \dots < x_N = 1$.

s.t $x_i - x_{i-1} = \frac{1}{N}$. Since

$$U_p(f) = \sum_{i=1}^{N} M_i (x_i - x_{i-1}) = \frac{1}{N} \text{ and } L_p(f) = 0,$$

we can conclude that $U_p(f) - L_p(f) = \frac{1}{N} < \varepsilon$.

Hence f is Riemann integrable. Since $\int_a^b f(x)\,dx = \sup_P \{L_p(f)\}$,

and $L_p(f) = 0$ for any partition P on $[0,1]$, we can conclude

that $\int_a^b f(x)\,dx = 0$.

4. Let $f(x) = 3x$ on $[0,1]$ and let $\varepsilon > 0$ be given.

b) Compute $\int_0^1 f(x)\,dx$ without using the Fundamental Theorem of Calculus.

For each $k \in \mathbb{N}$, we let P_k be a partition on $[0,1]$:
$0 = x_0 < x_1 < x_2 < \dots < x_k = 1$ such that $x_i - x_{i-1} = \frac{1}{k}$ for all $1 \le i \le k$.

Also, we set $S_k = \sum_{i=1}^{k} f(x_i)(x_i - x_{i-1})$.

Hence, $\{P_k\}_{k \in \mathbb{N}}$ is a sequence of partitions such that the maximum length of the subintervals goes to zero as $k \to \infty$.

Note $x_i = \frac{i}{k}$

Since
$$S_k = \sum_{i=1}^{k} f(x_i)(x_i - x_{i-1}) = \frac{1}{k}\sum_{i=1}^{k} 3x_i$$
$$= \frac{3}{k}\sum_{i=1}^{k}\frac{i}{k}$$
$$= \frac{3}{k^2}\left(\frac{k}{2}\right)(k+1)$$
$$= \frac{3}{2}\left(1 + \frac{1}{k}\right),$$

we can conclude that $\int_0^1 3x\,dx = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{3}{2}\left(1+\frac{1}{k}\right) = 3/2$.