Section 3.2

2) Prove that all cubic polynomials have at least one real root.

Proof Note: Let $p(x) = ax^3 + bx^2 + cx + d$ and $q(x) = x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$

α is a root of $p(x)$ iff α is a root of $q(x)$

So, to show that all cubic polynomials have at least one real root, it is enough to show that all cubic polynomials of the form $x^3 + a_2x^2 + a_1x + a_0$ have at least one real root.

Set $r(x) = x^3 + a_2x^2 + a_1x + a_0$.

Notice that $\lim_{n\to\infty} r(n) = \lim_{n\to\infty} n^3 + a_2n^2 + a_1n + a_0 = \infty$

and $\lim_{n\to\infty} r(-n) = \lim_{n\to\infty} -n^3 + a_2n^2 - a_1n + a_0 = -\infty$.

Hence $\exists N_1 \in \mathbb{N}$ s.t $r(n) \geq 1$ for all $n \geq N_1$

and $\exists N_2 \in \mathbb{N}$ s.t $r(m) \leq -1$ for all $m \geq N_2$.

By the intermediate value Thm, $\exists c \in \mathbb{R}$ s.t $r(c) = 0$.

So, c is a root of the polynomial $x^3 + a_2x^2 + a_1x + a_0$.

5. Let f and g be continuous function on $[a, b]$

Suppose that $f(x) < g(x)$ for all x in $[a, b]$

Prove that there is an $\alpha < 1$ s.t $f(x) \leq \alpha g(x)$ for all $x \in [a, b]$.

Proof Assume that there is no real number α between 0 and 1 s.t

$f(x) \leq \alpha g(x)$.

Hence for $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ s.t $f(x_n) > (1 - \frac{1}{n}) g(x_n)$. $(*)$

Since $\{x_n\}$ is a seq. in $[a, b]$, by Cor 2.6.3, there is a convergence subseq x_{n_k} s.t $\lim\limits_{k \to \infty} x_{n_k} = c$ is in $[a, b]$.

Since f and g are cont on $[a, b]$ and $c \in [a, b]$, we can conclude that

$$f(c) = \lim_{k \to \infty} f(x_{n_k}) \geq \lim_{k \to \infty} \left(1 - \frac{1}{n_k}\right) g(x_{n_k}) \quad (\text{by } (*))$$

$$= \lim_{k \to \infty} \left(1 - \frac{1}{n_k}\right) \lim_{k \to \infty} g(x_{n_k})$$

$= g(c)$. This contradicts with the fact that $f(x) < g(x)$ $\forall x \in [a, b]$. Hence, $\exists \alpha \in (0, 1)$ s.t $f(x) \leq \alpha g(x)$ $\forall x \in [a, b]$.

8 Prove that if f is Lipschitz continuous on a set $S \subseteq \mathbb{R}$ then f is uniformly continuous on S.

Proof Assume that there exists $M \in \mathbb{R}$ s.t

$$\frac{|f(x)-f(c)|}{|x-c|} \le M \quad \text{for all } x, c \in S \text{ s.t } x \neq c.$$

Let $\varepsilon > 0$. Set $\delta = \varepsilon/M$.

Then if $|x-c| < \delta$ then $|f(x)-f(c)| \le M|x-c| \le M\delta = M\varepsilon/M = \varepsilon$

Hence, f is uniformly cont. on S.

9 Show that the function $f(x) = \frac{1}{x}$ is not uniformly cont on $(0, \infty)$ but is uniformly cont on any interval of the form $[\mu, \infty)$ if $\mu > 0$.

Proof Set $\varepsilon < 1$. Let $\delta > 0$. By AP, $\exists n \in \mathbb{N}$ s.t $2n > \frac{1}{\delta}$ $\left(\Rightarrow \delta > \frac{1}{2n}\right)$

We choose $x = \frac{1}{n}$, $y = \frac{1}{2n}$.

We then have $|x-y| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \left|\frac{2n-n}{n\,2n}\right| = \frac{1}{2n} < \delta$

and $|f(x)-f(y)| = |n-2n| = n > \varepsilon$.

So, f is not uniformly cont ~~at~~ on $(0,\infty)$.

Next, we will show that f is uniformly cont on $[\mu, \infty)$ if $\mu > 0$.

Let $\varepsilon > 0$. Set $\delta = \mu^2 \varepsilon$

If $|x-y| < \delta$ then $|f(x) - f(y)|$

$$= \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \frac{|y-x|}{|xy|}$$

$$= \frac{|y-x|}{xy} \quad \text{since } x \geq \mu > 0 \text{ and } y \geq \mu > 0$$

$$\leq \frac{|y-x|}{\mu^2} \quad \text{since } x \geq \mu \text{ and } y \geq \mu,\ \mu > 0$$

$$\leq \frac{1}{\mu^2} \mu^2 \varepsilon = \varepsilon.$$

Hence, f is uniformly cont on $[\mu, \infty)$.