Section 3.1

2) Prove part (c) of Theorem 3.1.1: Let f and g be continuous functions and define $D = Dom(f) \cap Dom(g)$. Then fg is continuous on D.

Proof Note: fg is continuous on D iff fg is continuous at x for every $x \in D$.

Let $x \in D$ and let $\{x_n\}$ be a sequence in D s.t. $\lim_{n\to\infty} x_n = x$.

Goal: WTS $\lim_{n\to\infty} (fg)(x_n) = fg(x)$.

Since f and g are cont at x, it follows that

$\lim_{n\to\infty} f(x_n) = f(x)$ and $\lim_{n\to\infty} g(x_n) = g(x)$.

Since $\lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} f(x_n)g(x_n) = \lim_{n\to\infty} f(x_n) \lim_{n\to\infty} g(x_n) = f(x)g(x)$
$= (fg)(x)$,

it implies that fg is cont at x. Since x is an arbitrary elt in D, we can conclude that fg is cont on D.

5. Suppose that f is a continuous function on $\mathbb{R}$ s.t $f(q)=0$ for all $q \in \mathbb{Q}$. Prove that $f(x)=0$ for all $x \in \mathbb{R}$.

Proof To show that $f(x)=0$ for all $x \in \mathbb{R}$, it is enough to show that $f(p)=0$ for all irrational number p

Assume that $f(p)=b \neq 0$. Since f is continuous at p, we can conclude that with $\varepsilon = \frac{|b|}{2}$, there is $\delta > 0$ s.t

$$|x-p|<\delta \longrightarrow |f(x)-f(p)| < \frac{|b|}{2}$$

Let $q \in (p-\delta, p+\delta)$ s.t $q \in \mathbb{Q}$. Hence $|q-p|<\delta$. Moreover, $|f(q)-f(p)| = |0-b| = |b| < \frac{|b|}{2}$. This is a contradiction. Hence $f(p)=0$.
Consequently, $f(x)=0 \ \forall x \in \mathbb{R}$

11). Let $f(x)=\sqrt{x}$ with domain $\{x \mid x \geq 0\}$.
a) Let $\varepsilon > 0$ be given. For each $c>0$, show how to choose δ so that $|x-c| \leq \delta$ implies $|\sqrt{x}-\sqrt{c}| \leq \varepsilon$

$$|\sqrt{x}-\sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{|x-c|}{\sqrt{c}} \leq \frac{\delta}{\sqrt{c}} \leq \varepsilon.$$

Set $\delta = \sqrt{c}\,\varepsilon$.

b) Give a separate argument to show that f is continuous at zero.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$.

Then if $|x - 0| < \delta$ then $|f(x) - f(0)| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$.

So, f is cont at zero.

13) b) Find sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of numbers in $(0, 1]$, so that $\alpha_n \to 0$, $\beta_n \to 0$ and

$$\lim_{n\to\infty} \sin \frac{1}{\alpha_n} = 1, \quad \lim_{n\to\infty} \sin \frac{1}{\beta_n} = -1.$$

Sol^{n}. Set $\alpha_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$

So, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \sin \frac{1}{\alpha_n} = \lim_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$

Next, we set $\beta_n = \dfrac{1}{2n\pi - \pi/2}$

So, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \sin \frac{1}{\beta_n} = \lim_{n\to\infty} \sin(2n\pi - \frac{\pi}{2}) = -1$.