Problems for Sections 3.1-3.2.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 3x + 5$. Use $\epsilon - \delta$ definition to show that f is continuous at 2.
2. Use $\epsilon - \delta$ definition to show that $g(x) := \sqrt{x}$ is continuous on $[0, 2]$.
3. Prove that $g(x) := \sqrt{x}$ is uniformly continuous on $[1, \infty)$.
4. Assuming that $\cos x$ is continuous function, prove that $x = \cos x$ for some x in $(0, \pi/2)$.
5. Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. Hint: consider $g(x) = f(x) - f(x + \frac{1}{2})$.

1) Sol^n Let $\varepsilon > 0$. Set $\delta = \min\{1, \frac{\varepsilon}{2}\}$

If $|x-2| \leq \delta$ then $-\delta \leq x-2 \leq \delta$ and $-\delta+1 \leq x-1 \leq \delta+1$.

Consequently we have $|x-1| < 2$ and

$$|f(x) - f(2)| = |x^2 - 3x + 5 - (4 - 6 + 5)|$$
$$= |x^2 - 3x + 2|$$
$$= |x-2||x-1| \leq 2\frac{\varepsilon}{2} = \varepsilon.$$

So, f is cont at 2.

2) Sol^n Step 1 We will show that $f(x) = \sqrt{x}$ is cont at c where $c \in (0, 2]$.

Let $\varepsilon > 0$ Set $\delta = \sqrt{c}\,\varepsilon$

If $|x-c| < \delta$ then

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|\sqrt{x} - \sqrt{c}|\,|\sqrt{x} + \sqrt{c}|}{|\sqrt{x} + \sqrt{c}|}$$
$$= \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{|x-c|}{\sqrt{c}} \leq \frac{\sqrt{c}\,\varepsilon}{\sqrt{c}} = \varepsilon.$$

Hence, f is cont at c.

Step 2 We will show that f is cont at 0.

Let $\varepsilon > 0$. Set $\delta = \varepsilon^2$.

If $|x-0| < \delta$ then $|f(x) - f(0)| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$.

So, f is cont at 0.

3) Proof By 2), we know that $g(x) = \sqrt{x}$ is cont on $[1,2]$.

Hence $g(x) = \sqrt{x}$ is uniformly cont on $[1,2]$. ①

We will show that $g(x) = \sqrt{x}$ is uniformly cont on $[2, \infty]$.

Let $\varepsilon > 0$. Set $\delta = 2\sqrt{2}\,\varepsilon$

If $|x-y| < \delta$ then $|g(x) - g(y)|$

$$= |\sqrt{x} - \sqrt{y}|$$

$$= \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$$

$$\leq \frac{|x-y|}{2\sqrt{2}} \qquad \text{since } \sqrt{x} \geq \sqrt{2} \text{ and } \sqrt{y} \geq \sqrt{2}$$

$$\leq \frac{2\sqrt{2}\,\varepsilon}{2\sqrt{2}} = \varepsilon.$$

Hence g is uniformly cont on $[2, \infty)$ ②.

By ① & ②, we can conclude that g is uniformly cont on $[1, \infty)$.

4) Proof Set $f(x) = x - \cos x$.

Since $f(0) = 0 - \cos 0 = -1$ and $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2})$

by IMV, $\exists\, c \in (0, \frac{\pi}{2})$ s.t $f(c) = 0$.

Hence, $\exists\, c \in (0, \frac{\pi}{2})$ s.t $c - \cos c = 0$.

5) Proof Set $g(x) = f(x) - f(x + \frac{1}{2})$.

Since $g(0) = f(0) - f(\frac{1}{2})$ and $g(\frac{1}{2}) = f(\frac{1}{2}) - f(1)$

$$= f(\tfrac{1}{2}) - f(0)$$

$$= -g(0),$$

by IMV, $\exists\, c \in (0, \frac{1}{2})$ s.t $g(c) = 0$. Hence $f(c) = f(c + \frac{1}{2})$.