Solution for Section 2.6 problems

3. Suppose that the sequence $\{a_n\}$ converges to a and that d is a limit point of the sequence $\{b_n\}$. Prove that ad is a limit point of the sequence $\{a_nb_n\}$.

Proof. Since d is a limit point of the sequence $\{b_n\}$, it follows that there is a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ which converges to d. Since $\{a_n\}$ converges to a, it follows that a subsequence $\{a_{n_k}\}$ also converges to a.

Now, we will show that ad is the limit of a subsequence $\{a_{n_k}b_{n_k}\}$ of $\{a_nb_n\}$. Since

$$\lim_{k\to\infty} a_{n_k}b_{n_k} = \lim_{k\to\infty} a_{n_k} \cdot \lim_{k\to\infty} b_{n_k} = ad,$$

it implies that ad is the limit of a subsequence $\{a_{n_k}b_{n_k}\}$ of $\{a_nb_n\}$. Hence ad is a limit point of the sequence $\{a_nb_n\}$.

□

4. Let c be a limit point of $\{a_n\}$ and d be a limit point of $\{b_n\}$. Is $c+d$ necessarily a limit point of $\{a_n+b_n\}$? Prove it or give a counterexample.

Counterexample Set $a_n = (-1)^n$ and $b_n = (-3)^n$ for all $n \in \mathbb{N}$. Clearly, -1 is a limit point of $\{a_n\}$ and 3 is a limit point of $\{b_n\}$. Notice that $\{a_n+b_n\} = \{(-4)^n\}$ and $-1+3$ is not a limit point of $\{a_n+b_n\}$.

8. Let $\{I_k\}_{k=1}^{\infty}$ be a nested family of closed, finite intervals; that is $I_1 \supseteq I_2 \supseteq$ .... Prove that there is a point p contained in all intervals, that is, $p \in \cap_{k=1}^{\infty} I_k$.

Proof. We will assume that $I_k = [a_k, b_k]$ for $k \in \mathbb{N}$. Since $\{I_k\}_{k=1}^{\infty}$ is a nested family, we can conclude that

$$a_1 \le a_2 \le ... \le a_k \le a_{k+1} \le ..... \quad (0.1)$$

$$.... \le b_{k+1} \le b_k \le ... \le b_2 \le b_1 \quad (0.2)$$

Set $A = \{\ a_k \mid k \in \mathbb{N}\}$. We will show that b_k is an upper bound of A for all $k \in \mathbb{N}$. Let $i \in \mathbb{N}$. Since $a_1 \le a_2 \le ... \le a_i$ and $a_i < b_i$, we can conclude that $a_j \le b_i$ for all $j \in \{1, 2, ..., i\}$. Let $j \in \mathbb{N}$ such that $j > i$. Since $a_j < b_j$ and $b_j \le b_i$ (cf. 0.2), these imply that $a_j < b_i$. Therefore, $a_j \le b_i$ for all $j \in \mathbb{N}$ and b_i is an upper bound of A. Since i is an arbitrary positive integer, we can conclude that b_i is an upper bound of A for all $i \in \mathbb{N}$. Consequently, $Sup\ A$ exists.

We will denote the supremum of A by p. Clearly, $a_k \le p \le b_k$ for all $k \in \mathbb{N}$. Hence, $p \in I_k$ for all $k \in \mathbb{N}$. This implies that $p \in \cap_{k=1}^{\infty} I_k$. □

9. Suppose that $\{x_n\}$ is a monotone increasing sequence of points in $\mathbb{R}$ and suppose that a subsequence of $\{x_n\}$ converges to a finite limit. Prove that $\{x_n\}$ converges to a finite limit.

Proof. Let $\{x_n\}$ be a monotone increasing sequence and let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$. Therefore, there exists $a \in \mathbb{R}$ such that $x_{n_k} \leq a$ for all $k \in \mathbb{N}$.

Assume that $\{x_n\}$ is an unbounded sequence. This implies that the set $\{x_n | n \in \mathbb{N}\}$ is unbounded above. Since a is not an upper bound of $\{x_n | n \in \mathbb{N}\}$, it follows that there exists $N \in \mathbb{N}$ such that $x_N > a$. Since $\{x_n\}$ is an increasing sequence, we can conclude that

$$x_n > a \text{ for all } n \geq N. \tag{0.3}$$

Since $n_k \to \infty$ as $k \to \infty$, it follow that there exists $K \in \mathbb{N}$ such that

$$n_k \geq N \text{ for all } k \geq K. \tag{0.4}$$

By equations (0.3)-(0.4), we then have that $x_{n_k} > a$ for all $k \geq K$. This is a contradiction. Therefore, $\{x_n\}$ is bounded. Since $\{x_n\}$ is a monotone bounded sequence, it implies that $\{x_n\}$ is a convergent sequence. □