Solution for Section 2.5 problems

2. Let S be a set of real numbers and suppose that a_n is an upper bound for S for each n. Prove that if $a_n \to a$, then a is an upper bound for S.

Proof. We will proof by contradiction. Assume that a is not an upper bound of S. Then there exists $s \in S$ such that $a < s$.

Set $\epsilon = s - a > 0$. Since $\lim_{n\to\infty} a_n = a$, we can conclude that there exists $N \in \mathbb{N}$ such that $|a_n - a| \leq s - a$ for all $n \geq N$. Since $a_n - a \leq |a_n - a| \leq s - a$ for all $n \geq N$, it follows that $a_n \leq s$ for all $n \geq N$. Since a_n is an upper bound of S for all $n \in \mathbb{N}$, so we have that $a_n = s$ for all $n \geq N$. Therefore, $\lim_{n\to\infty} a_n = s$, and $s = a$. This contradicts with the fact that $a < s$. Hence, a is an upper bound of S. □

3. Suppose that a set S of real numbers is bounded and let μ be an upper bound for S. Show that μ is the least upper bound of S if and only if for every $\epsilon > 0$ there is an element of S in the interval $[\mu - \epsilon, \mu]$.

Proof. ($\rightarrow$) Assume that μ is the least upper bound of S. Let ϵ be a real number that is greater than zero. Since $\mu - \epsilon < \mu$ and μ is a least upper bound of S, we can conclude that $\mu - \epsilon$ is not an upper bound of S. This implies that there is $s \in S$ such that $\mu - \epsilon < s \leq \mu$. Hence, $s \in [\mu - \epsilon, \mu]$.

($\leftarrow$) Assume that for every $\epsilon > 0$ there is an element of S in the interval $[\mu - \epsilon, \mu]$. Let v be a real number such that $v < \mu$. We will show that v is not an upper bound of S. Set $\epsilon = \mu - v > 0$. By the assumption, there is an element $s' \in S$ such that $s' \in [\mu - \epsilon, \mu]$. Since $v = \mu - \epsilon$ and $\mu - \epsilon \leq s' \leq \mu$, it implies that v is not an upper bound of S. We can conclude further that μ is a least upper bound of S. □

4. Prove that least upper bounds are unique. That is, if μ_1 and μ_2 are both least upper bounds for the set S then $\mu_1 = \mu_2$.

Proof. Let μ_1 and μ_2 be both least upper bounds for the set S. Since μ_1 is a least upper bound of S and μ_2 is an upper bound of S, it follows that

$$\mu_1 \leq \mu_2. \tag{0.1}$$

By using similar arguments, we can conclude that

$$\mu_2 \leq \mu_1. \tag{0.2}$$

. By equations (0.1)-(0.2),we then have that $\mu_1 = \mu_2$. □

7. Show that if $sup\ S = +\infty$, then there is a sequence of points $s_n \in S$ such that $s_n \to +\infty$.

Proof. Since 1 is not an upper bound of S, it follows that there is $s_1 \in S$ such that $s_1 \geq 1$. Set $n_2 = max\{2, s_1\}$. Since n_2 is not an upper bound of S, it follows that there is $s_2 \in S$ such that $s_2 \geq n_2$. Hence, $s_2 \geq s_1$ and $s_2 \geq 2$. Set $n_3 = max\{3, s_2\}$. Since n_3 is not an upper bound of S, we then have that there exists $s_3 \in S$ such that $s_3 \geq n_3$. Therefore, $s_3 \geq s_2 \geq s_1$ and $s_3 \geq 3$. Continue in this way, we are be able to construct an increasing sequence $\{s_n\}$ such that $s_n \geq n$ for all $n \in \mathbb{N}$.

We will show that $\{s_n\}$ diverges to $+\infty$. Let M be a positive real number. By Achimedian property, there is an $N \in \mathbb{N}$ such that $M \leq N$. Since $s_n \geq s_N \geq N \geq M$ for all $n \geq N$, it follows that $s_n \to \infty$. $\square$