Solution for Section 2.4 problems

4. Suppose that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Prove that $\lim_{n\to\infty}\{x_n - y_n\}$ exists.

Proof. Since $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences, it follows that $\{x_n\}$ and $\{y_n\}$ are both convergent sequences. Hence, we may assume that $x_n \to x$ and $y_n \to y$. This implies that $\lim_{n\to\infty}\{x_n - y_n\} = x - y$. So, $\lim_{n\to\infty}\{x_n - y_n\}$ exists. □

5. Suppose that $\{a_n\}$ is a Cauchy sequence. Prove that $\{a_n^2\}$ is a Cauchy sequence. Is the converse true?

Proof. Let $\{a_n\}$ be a Cauchy sequence. Hence, it is a convergent sequence and there exists $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$. It follows that

$$\lim_{n\to\infty} a_n^2 = \left(\lim_{n\to\infty} a_n\right)^2 = a^2.$$

So, $\{a_n^2\}$ is a convergent sequence which is a Cauchy sequence.

The converse is not true. Set $a_n = (-1)^n$ for all $n \in \mathbb{N}$. Clearly $\{a_n^2\} = \{1\}$ is a Cauchy sequence and $\{a_n\}$ is not a Cauchy sequence. □

7. Suppose that the terms $\{a_n\}$ satisfy $|a_{n+1} - a_n| \le 2^{-n}$ for all n. Prove that $\{a_n\}$ is a Cauchy sequence.

Proof. Let $\epsilon > 0$. By Archimedian property, there exists $N \in \mathbb{N}$ such that $2^{1-N} \le \epsilon$.

Let $m, n \in \mathbb{N}$ such that $m \ge n \ge N$. We may assume that $m = n + k$ for some $k \in \mathbb{N}$. Since

$$\begin{aligned}
|a_m - a_n| &= |a_{n+k} - a_n| \\
&= |a_{n+k} - a_{n+k-1} + a_{n+k-1} - .... - a_{n+1} + a_{n+1} - a_n| \\
&\le |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + ... + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n| \\
&\le \frac{1}{2^{n+k-1}} + \frac{1}{2^{n+k-2}} + ... + \frac{1}{2^{n+1}} + \frac{1}{2^n} \\
&\le \frac{1}{2^n}\left(1 + \frac{1}{2} + ... + \frac{1}{2^{k-1}}\right) \\
&\le \frac{1}{2^N}\frac{1}{1-\frac{1}{2}} = \frac{2}{2^N} \le \epsilon,
\end{aligned}$$

we can conclude that $\{a_n\}$ is a Cauchy sequence. □

10. Let $a_1 = \sqrt{2}$, and let a_n for $n \ge 2$ be defined recursively by the formula

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}.$$

a) Prove by induction that $\sqrt{2} \le a_n \le 2$ for all n.

Proof. Clearly, $\sqrt{2} \le a_1 \le 2$. We now assume that $\sqrt{2} \le a_k \le 2$ for some $k \in \mathbb{N}$. We will show that $\sqrt{2} \le a_{k+1} \le 2$. Since $0 \le a_k \le 4$, we then have that $0 \le \sqrt{a_k} \le 2$. Hence, $2 \le 2 + \sqrt{a_k} \le 4$. This implies that $\sqrt{2} \le a_{k+1} = \sqrt{2 + \sqrt{a_k}} \le 2$.

By induction, we can conclude that $\sqrt{2} \le a_n \le 2$ for all n. □

b) Prove that $\{a_n\}$ is a Cauchy sequence and conclude that $\{a_n\}$ converges.

Proof. Claim $a_{n+1} \ge a_n$ for all $n \in \mathbb{N}$.

We will use an induction method to prove the above statement. Clearly, $a_2 = \sqrt{2 + \sqrt{2}} \ge \sqrt{2} = a_1$. Hence, the statement is true for $n = 1$.

Now, we assume that $a_{k+1} \ge a_k$ for some $k \in \mathbb{N}$. This implies that

$$a_{k+2} = \sqrt{2 + \sqrt{a_{k+1}}} \ge \sqrt{2 + \sqrt{a_k}} = a_{k+1}.$$

By an induction method, we can conclude that $a_{n+1} \ge a_n$ for all $n \in \mathbb{N}$.

By a) and the claim, we know that $\{a_n\}$ is a bounded increasing sequence. Hence, $\{a_n\}$ is a convergence sequence which implies that it is a Cauchy sequence. □