Solution for Section 2.2 problems

1(c) Prove that the limit of the sequence $a_n = \frac{n^2+6}{3n^2-2}$ exists by using Theorem 2.2.3-Theorem 2.2.6.

Proof.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+6}{3n^2-2} = \lim_{n\to\infty} \frac{(n^2+6)/n^2}{(3n^2-2)/n^2} = \lim_{n\to\infty} \frac{1+\frac{6}{n^2}}{3-\frac{2}{n^2}} = 1/3.$$

□

2(a) Find the limit of the sequence $\{a_n\} = \{e^{-n}\sin n\}$.

Proof. Since $-1 \le \sin n \le 1$ for all $n \in \mathbb{N}$, we then have that

$$-\frac{1}{e^n} \le \frac{\sin n}{e^n} \le \frac{1}{e^n} \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} -e^{-n} = \lim_{n\to\infty} e^{-n} = 0$, we can conclude that

$$\lim_{n\to\infty} \frac{\sin n}{e^n} = 0.$$

□

3. Prove Theorem 2.2.4: let $\{a_n\}$ be a sequence and suppose that $a_n \to a$. Then, for any constant k, $lim_{n\to\infty}(ka_n) = k\lim_{n\to\infty} a_n = ka$.

Proof. Let $\{a_n\}$ be a convergent sequence such that $\lim_{n\to\infty} a_n = a$. Let k be a constant number.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, it follows that there exists $N \in \mathbb{N}$ such that $|a_n - a| \le \frac{\epsilon}{|k|+1}$ for all $n \ge N$. So, for any $n \ge N$, we have

$$|ka_n - ka| = |k||a_n - a| \le |k|\left(\frac{\epsilon}{|k|+1}\right).$$

Therefore, $\lim_{n\to\infty} ka_n = ka$. □

5. Prove Theorem 2.2.6: let $\{a_n\}$ and $\{b_n\}$ be sequences and suppose that $a_n \to a$ and $b_n \to b$. Suppose that $b \neq 0$ and $b_n \neq 0$ for any n. Then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$.

Proof. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences such that $a_n \to a$ and $b_n \to b$. Suppose that $b \neq 0$ and $b_n \neq 0$ for any n.

Claim: $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Since $\lim_{n\to\infty} b_n = b$, we conclude that there exists $N_1 \in \mathbb{N}$ such that $|b_n - b| \leq \frac{|b|}{2}$ for all $n \geq N_1$. Since $||b_n| - |b|| \leq |b_n - b| \leq \frac{|b|}{2}$ for all $n \geq N_1$, it implies that

$$-\frac{|b|}{2} \leq |b_n| - |b| \leq \frac{|b|}{2} \text{ for all } n \geq N_1.$$

Hence, $\frac{|b|}{2} \leq |b_n| \leq \frac{3}{2}|b|$, and $\frac{1}{|b_n|} \leq \frac{2}{|b|}$ for all $n \geq N_1$.

Let $\epsilon > 0$. There exists $N_2 \in \mathbb{N}$ such that $|b_n - b| \leq \epsilon\frac{|b|^2}{2}$ for all $n \geq N_2$. Set $N = max\{N_1, N_2\}$. We then have

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|b_n||b|} \leq \frac{2}{|b|^2}\epsilon\frac{|b|^2}{2} = \epsilon \text{ for all } n \geq N.$$

Consequently, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Now, we will show that $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$.

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} \frac{1}{b_n}\right) = a \cdot \frac{1}{b}.$$

□

6. Let $p(x)$ be any polynomial and suppose that $a_n \to a$. Prove that

$$\lim_{n\to\infty} p(a_n) = p(a).$$

Proof. Let $p(x) = c_i x^i + c_{i-1}x^{i-1} + ... + c_1 x + c_0$ be a polynomial and let $\{a_n\}$ be a convergent sequence such that $\lim_{n\to\infty} a_n = a$.

So, we have

$$\begin{aligned}
\lim_{n\to\infty} p(a_n) &= \lim_{n\to\infty} c_i(a_n)^i + c_{i-1}(a_n)^{i-1} + ... + c_1 a_n + c_0 \\
&= \lim_{n\to\infty} c_i(a_n)^i + \lim_{n\to\infty} c_{i-1}(a_n)^{i-1} + ... + \lim_{n\to\infty} c_1 a_n + \lim_{n\to\infty} c_0 \\
&= c_i \lim_{n\to\infty} (a_n)^i + c_{i-1} \lim_{n\to\infty} (a_n)^{i-1} + ... + c_1 \lim_{n\to\infty} a_n + \lim_{n\to\infty} c_0 \\
&= c_i a^i + c_{i-1}a^{i-1} + ... + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

□