Solution for Section 2.1 problems

2(c). Prove directly that the sequence $a_n = 3 + 2^{-n}$ converges by letting $\epsilon > 0$ be given and finding $N(\epsilon)$ so that (1) holds.

Proof. Let $\epsilon > 0$. By Archimedian Property, there exists $N \in \mathbb{N}$ such that $N \geq \frac{\ln(1/\epsilon)}{\ln 2}$. Hence, for $n \geq N$, we have $n \geq \frac{\ln(1/\epsilon)}{\ln 2}$ and $(1/2^n) \leq \epsilon$.

Since $|a_n - 3| = |3 + \frac{1}{2^n} - 3| = |\frac{1}{2^n}| \leq \epsilon$ for all $n \geq N$, we can conclude that $\lim_{n\to\infty} 3 + \frac{1}{2^n} = 3$. □

4(c). Prove directly that the sequence $a_n = \sqrt{\ln n}$ diverges to ∞.

Proof. Let $M \in \mathbb{R}$. By Archimedian Property, there exists $N \in \mathbb{N}$ such that $N \geq e^{M^2}$. Hence for all $n \geq N$, we have $n \geq e^{M^2}$ and $a_n = \sqrt{\ln n} \geq M$. Since M is an arbitrary real number, it follows that $\sqrt{\ln n} \to \infty$. □

5. Prove that a sequence $\{a_n\}$ can have at most one limit.

Proof. Let $\{a_n\}$ be a convergent sequence. Let a and b be limits of the sequence $\{a_n\}$. Assume that $a \neq b$. Set $\epsilon = \frac{|b-a|}{3} > 0$. Since $\{a_n\}$ converges to a, it implies that there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| \leq \frac{|b-a|}{3} \text{ for all } n \geq N_1.$$

Since $\{a_n\}$ converges to b, it follows that there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - b| \leq \frac{|b-a|}{3} \text{ for all } n \geq N_2.$$

Set $N = max\{N_1, N_2\}$. We have

$$|a-b| = |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b| \leq \frac{|b-a|}{3} + \frac{|b-a|}{3} \text{ for all } n \geq N.$$

This is a contradiction. Hence, $a = b$. □

8. Suppose that $a_n \to 0$ and $b_n \to \infty$. Show that we can not draw any definite conclusions about the sequence $c_n = a_n b_n$ by giving examples which satisfy these hypotheses and

a) $c_n \to 0$.
$a_n = \frac{1}{n^2}$, $b_n = n$ and $c_n = \frac{1}{n}$.

b) $c_n \to \infty$.
$a_n = \frac{1}{n}$, $b_n = n^2$ and $c_n = n$.

c) $c_n = 1$.
$a_n = \frac{1}{n}$, $b_n = n$ and $c_n = 1$.

d) $\{c_n\}$ does not converge, nor does it diverge to $+\infty$ or $-\infty$.
$a_n = \frac{(-1)^n}{n}$, $b_n = n$ and $c_n = (-1)^n$.