

Solution for Chapter 1: Problems

Label each statement as true or false. If a statement is true, prove it. If not,

1. give an example of why it is false, and
 2. if possible, correct it to make it true, and then prove it.
1. If $\lim_{n \rightarrow \infty} \frac{a_n}{n} = A$ with $A \neq 0$ a constant, then the sequence $\{a_n\}$ diverges to infinity.

False. Let $a_n = -n$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{n} = -1$ and the sequence $\{a_n\}$ diverges to negative infinity.

2. If the sequence $\{a_n\}$ is bounded, then it is Cauchy.

False. $\{(-1)^n\}$ is bounded and it is not Cauchy since it is a divergent sequence.

3. The sequence $\{a_n\}$ with $a_n = \frac{2^n}{1+2^n}$ is monotone.

True since $a_{n+1} = \frac{2^{n+1}}{1+2^{n+1}} = \frac{2 \cdot 2^n}{1+2 \cdot 2^n} > \frac{2 \cdot 2^n}{2+2 \cdot 2^n} = \frac{2^n}{1+2^n} = a_n$.

4. The sequence $\{a_n\}$ converges if and only if it is monotone.

False. A sequence $\{n\}$ is monotone and it is a divergent sequence.

5. Suppose that the sequence $\{a_n\}$ converges to A . Define a new sequence $\{b_n\}$ by $b_n = \frac{2}{3}a_n + \frac{1}{3}a_{n+2}$ for all $n \in \mathbb{N}$. Then $\{b_n\}$ converges to A .

True since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{2}{3}a_n + \frac{1}{3}a_{n+2} = \frac{2}{3} \lim_{n \rightarrow \infty} a_n + \frac{1}{3} \lim_{n \rightarrow \infty} a_{n+2} = (2/3)A + (1/3)A = A$.

6. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences such that $\{a_n + b_n\}$ converges to A and $\{a_n - b_n\}$ converges to B . Then the sequence $\{a_n b_n\}$ converges to $\frac{1}{4}(A^2 - B^2)$.

True. Since $a_n b_n = \frac{1}{4}((a_n + b_n)^2 - (a_n - b_n)^2)$, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n b_n &= \frac{1}{4} \left(\lim_{n \rightarrow \infty} (a_n + b_n)^2 - \lim_{n \rightarrow \infty} (a_n - b_n)^2 \right) \\ &= \frac{1}{4} \left(\left(\lim_{n \rightarrow \infty} a_n + b_n \right)^2 - \left(\lim_{n \rightarrow \infty} a_n - b_n \right)^2 \right) \\ &= \frac{1}{4} (A^2 - B^2). \end{aligned}$$

7. Suppose that $\{a_n\}, \{b_n\}$ are two sequences such that $\{a_n\}$ and $\{\frac{a_n}{b_n}\}$ both converge. Then $\{\frac{b_n}{a_n}\}$ or $\{b_n\}$ must converge.

False. Set $a_n = 1$ and $b_n = n$ for all $n \in \mathbb{N}$. Then $\{a_n\}$ and $\{\frac{a_n}{b_n}\}$ both converge. However, $\{\frac{b_n}{a_n}\}$ or $\{b_n\}$ both diverge.

8. If S is a set of real numbers and $A = \text{Sup } S$, then $A - \epsilon$ must be in S for any $\epsilon > 0$.

False. Set $S = \{1, 2, 3\}$ Clearly, $A = 3$. Set $\epsilon = \frac{1}{2}$. Then $A - \epsilon = 5/2$ is not in S .

9. If $a_n \neq 0$ for any $n \in \mathbb{N}$, and $\{a_n\}$ converges to 0 then $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$.

False. Let $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n \neq 0$ and $a_n \rightarrow 0$. However, $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = 1$.

10. The sequence $\{a_n\}$ converges to zero if and only if $\{|a_n|\}$ converges to zero.

True

Proof. (\rightarrow) Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - 0| \leq \epsilon$ for all $n \geq N$. This implies that $||a_n| - 0| = |a_n| \leq \epsilon$ for all $n \geq N$ and $\lim_{n \rightarrow \infty} |a_n| = 0$.

(\leftarrow) Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $||a_n| - 0| \leq \epsilon$ for all $n \geq N$. This implies that $|a_n - 0| = ||a_n|| \leq \epsilon$ for all $n \geq N$ and $\lim_{n \rightarrow \infty} a_n = 0$. \square

11. If $\{a_n\}$ and $\{b_n\}$ are two sequences, where $\{a_n\}$ converges to zero and $\{b_n\}$ is bounded then the sequence $\{a_n b_n\}$ converges to zero.

True

Proof. Since $\{b_n\}$ is bounded, it implies that there exists a positive real number M such $|b_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Since $a_n \rightarrow 0$, it follows that there exists $N \in \mathbb{N}$ such that $|a_n - 0| \leq \epsilon/M$ for all $n \geq N$. Consequently, we have $|a_n b_n - 0| = |a_n| |b_n| \leq (\frac{\epsilon}{M}) M = \epsilon$ for all $n \geq N$. Hence, $a_n b_n \rightarrow 0$. \square