<u>Quiz 7</u>

1) Range of $T = \{ T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n \}$.

i) Since $T(\vec{0}) = \vec{0}$, we can conclude that $\vec{0} \in Range(T)$

ii) Closed under vector addition.

Let $T(\vec{a}), T(\vec{b}) \in Range(T)$.

Since T is a linear transformation and $\vec{a} + \vec{b} \in \mathbb{R}^n$, it follows that $T(\vec{a}) + T(\vec{b}) = T(\vec{a} + \vec{b}) \in Range(T)$

So, $Range(T)$ is closed under vector addition.

iii) Closed under scalar multiplication.

Let $T(\vec{a}) \in Range(T)$, $c \in \mathbb{R}$.

Since T is linear and $c\vec{a} \in \mathbb{R}^n$, we then have that $cT(\vec{a}) = T(c\vec{a}) \in Range(T)$. Hence, $Range(T)$ is closed under scalar multiplication.

By i) – iii) we can conclude that $Range(T)$ is a subspace of $\mathbb{R}^m$.

2) $V = Col\left(\begin{bmatrix} 1 & -1 & -3 & 1 \\ -1 & 1 & 3 & 2 \\ -3 & 1 & -1 & -1 \\ 2 & -2 & -6 & 1 \end{bmatrix} \right) = Span\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -1 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Note $\begin{bmatrix} 1 & -1 & -3 & 1 \\ -1 & 1 & 3 & 2 \\ -3 & 1 & -1 & -1 \\ 2 & -2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \left(-\frac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \in V$.

So, $V = Span\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$. Moreover, $\dim V = 3$.

Since $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ -3 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, it follows that

$\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is lin indep subset of V.

Since $\dim V = 3$, we can conclude that $\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
is a basis of V that contains $\alpha = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$