MAT 175 Fall 2009: Quiz 5
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Name:.............Solution...

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (5 points) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}.$$

Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$. Justify your answer.

<u>**Solution**</u>

Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, it follows that

$$\begin{aligned} & T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ = \ & x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ = \ & 2x_1 T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) + 5x_1 T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) + 3x_2 T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \\ = \ & (2x_1 + x_2)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (5x_1 + 3x_2)\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \\ = \ & \begin{bmatrix} 17x_1 + 10x_2 \\ -2x_1 - x_2 \\ -6x_1 - 4x_2 \end{bmatrix}. \end{aligned}$$

2. (5 points) Find a generating set for the range of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 4x_2 \\ 2x_1 - 3x_2 \\ 0 \\ x_2 \end{bmatrix}$. Please show your work in detail.

Solution Since

$$\begin{aligned}
& \text{Range of } T \\
= & \{T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2\} \\
= & \{\begin{bmatrix} x_1 - 4x_2 \\ 2x_1 - 3x_2 \\ 0 \\ x_2 \end{bmatrix} \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2\} \\
= & \{\begin{bmatrix} 1 & -4 \\ 2 & -3 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2\} \\
= & Span\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}\},
\end{aligned}$$

we can conclude that a generating set is $\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}\}$.