MAT 175 Fall 2009: Quiz 11
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Name: Solution

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (5 points) Let $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Find, if possible, an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Otherwise, explain why A is not diagonalizable.

2. (5 points) Let $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & c & 0 \\ 1 & 0 & -2 \end{bmatrix}$. Assume that the characteristic polynomial of B is $-(t-c)(t+2)(t-3)$. Determine all values of the scalar c for which the matrix B is not diagonalizable.

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$= (5-\lambda)\det\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - 4\det\begin{bmatrix} 4 & 2 \\ 2 & 2-\lambda \end{bmatrix} + 2\det\begin{bmatrix} 4 & 5-\lambda \\ 2 & 2 \end{bmatrix}$$

$$= (5-\lambda)\big((5-\lambda)(2-\lambda) - 4\big) - 4\big(4(2-\lambda) - 4\big) + 2\big(4(2) - 2(5-\lambda)\big)$$

$$= (5-\lambda)(6 - 7\lambda + \lambda^2) - 4(4 - 4\lambda) + 2(2\lambda - 2)$$

$$= (5-\lambda)(\lambda-1)(\lambda-6) + 16(\lambda-1) + 4(\lambda-1)$$

$$= (\lambda-1)\big((5-\lambda)(\lambda-6) + 20\big) = (\lambda-1)(-30 - \lambda^2 + 5\lambda + 6\lambda + 20)$$

$$= (\lambda-1)(-\lambda^2 + 11\lambda - 10) = -(\lambda-1)(\lambda-1)(\lambda-10)$$

$$= -(\lambda-1)^2(\lambda-10).$$

The eigenvalues are 1, and 10

$$E_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} -x_2 - \frac{1}{2} x_3 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_2, x_3 \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ Hence the multiplicity of 1 equals $\dim E_1 = 2$.

$$E_{10} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Hence, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$, $P = \begin{bmatrix} -1 & -1/2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

2) If $c \neq -2$ and 3 then B is diagonalizable.

<u>case1</u> $c=-2$. Then the characteristic polynomial of B is $-(t+2)^2(t-3)$.

$$E_{-2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{bmatrix} 3+2 & 0 & 0 \\ 0 & -2+2 & 0 \\ 1 & 0 & -2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{since} \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

x_2 and x_3 are free variables

Since the multiplicity of -2 equals $\dim E_{-2} = 2$, we can conclude that B is diagonalizable.

<u>case2</u> $c=3$. Then the characteristic polynomial of B is $-(t-3)^2(t+2)$.

$$E_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{since } x_1 = 5x_3$$

and x_2, x_3 are free variables.

Since the multiplicity of 3 equals $\dim E_3$, we can conclude that B is diagonalizable.

So, there is no values of c s.t. B is not diagonalizable.