2. Determine the values of r and s for which the given system of linear equations has

   (a) no solutions,

   (b) exactly one solution, and

   (c) infinitely many solutions.

   Please show your work in detail. (10 points)

$$\begin{aligned} x_1 + r x_2 &= -3 \\ 2x_1 + 5x_2 &= s. \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & r & -3 \\ 2 & 5 & s \end{array}\right] \quad R_2 - 2R_1 \mapsto R_2 \quad \left[\begin{array}{cc|c} 1 & r & -3 \\ 0 & 5-2r & s+6 \end{array}\right]$$

1) The given system has no solutions when $5-2r=0$ and $s+6 \neq 0$.
($\Rightarrow$ $r = 5/2$ and $s \neq -6$).

2) The given system has exactly one solution when $5-2r \neq 0$ (ie, $r \neq 5/2$).

3) The system has infinitely many solutions when $5-2r=0$ and $s+6=0$
(ie when $r = 5/2$ and $s = -6$).

4. Determine, if possible, a value of r for which the given set is linearly dependent. Please show your work in detail. (10 points)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \\ r \end{bmatrix} \right\}\}.$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ 3 & 6 & -2 \\ -1 & 1 & r \end{bmatrix} \xrightarrow[\substack{R_3 - 3R_1 \mapsto R_3 \\ R_4 + R_1 \mapsto R_4}]{R_2 - 2R_1 \mapsto R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 5 \\ 0 & -3 & 1 \\ 0 & 4 & r-1 \end{bmatrix} \xrightarrow[R_3 + R_1 \mapsto R_3]{R_2 / -5 \mapsto R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 4 & r-1 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & r-1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 4R_2 \mapsto R_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & r+3 \\ 0 & 0 & 0 \end{bmatrix}$$

The given set is linearly dependent when rank $A < 3$.

Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \\ r \end{bmatrix} \right\}$ is linearly dependent when $r = -3$.

5. (a) Determine if $\begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}$ is in $Span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$. Please show your work in detail. (8 points)

(b) Determine if $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is consistent for every $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$. (2 points)

a) Since $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 4 \\ 1 & 1 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$, we can conclude that

$\begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} \in Span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

b) Since $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, we can conclude that

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin Span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$. Furthermore,

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is not consistent for every $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 - 2x_3 \\ x_1 - x_3 \\ -x_1 + 2x_2 - 3x_3 \end{bmatrix}$.

Use the definition of linear transformation to prove that T is a linear transformation. **You will not receive any points if you use other methods.** (10 points)

1) We will show that T preserves vector addition

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \in \mathbb{R}^3$.

Since $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = T\left(\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}\right) = \begin{bmatrix} (b+e) - 2(c+f) \\ (a+d) - (c+f) \\ -(a+d) + 2(b+e) - 3(c+f) \end{bmatrix}$

$= \begin{bmatrix} b-2c \\ a-c \\ -a+2b-3c \end{bmatrix} + \begin{bmatrix} e-2f \\ d-f \\ -d+2e-3f \end{bmatrix}$

$= T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + T\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right)$,

we can conclude that T preserves vector addition.

2) We will show that T preserves scalar multiplication.

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ and let α be a scalar.

Since $T\left(\alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}\right) = \begin{bmatrix} (\alpha b) - 2(\alpha c) \\ (\alpha a) - (\alpha c) \\ -(\alpha a) + 2(\alpha b) - 3(\alpha c) \end{bmatrix}$

$= \alpha \begin{bmatrix} b-2c \\ a-c \\ -a+2b-3c \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$,

we can conclude that T preserves scalar multiplication.

By 1) and 2), T is a linear transformation.

MAT 175 Fall 2009: Exam 2
Dr. Gaywalee Yamskulna
11/06/09

Name:...

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (a) (2 points) Please give a definition of a linear transformation.

   (b) (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 \\ 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$.

   Use the definition of a linear transformation to show that T is a linear transformation. **You will not receive any points if you use other methods.**

   (c) (1 point) Determine whether T is one-to-one or not. Please show your work in detail.

   (d) (1 point) Determine whether T is onto or not. Please show your work in detail.

b) 1) T preserves vector addition.

Let $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$

Since $T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} 2(b+d) \\ 2(a+c) - (b+d) \\ (a+c)+(b+d) \end{bmatrix}$

$$= \begin{bmatrix} 2b \\ 2a - b \\ a + b \end{bmatrix} + \begin{bmatrix} 2d \\ 2c - d \\ c + d \end{bmatrix}$$

$$= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right),$$

it follows that T preserves vector addition.

2) T preserves scalar multiplication.

Let α be a scalar and let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

Since $T\left(\alpha \begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = \begin{bmatrix} 3(\alpha b) \\ 2(\alpha a) - \alpha b \\ \alpha a + \alpha b \end{bmatrix}$

$= \alpha \begin{bmatrix} 3b \\ 2a - b \\ a + b \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$,

we can conclude that T preserves scalar multi.

By 1) and 2), T is a linear transformation.

2. (a) (6 points) Find the determinant of the matrix $\begin{bmatrix} 2 & 2 & 4 & -3 \\ 0 & 0 & -4 & 0 \\ 6 & 0 & 1 & 5 \\ 0 & -5 & 2 & 0 \end{bmatrix}$. Please show your work in detail.

(b) (4 points) Determine the values of c for which the matrix $\begin{bmatrix} c & c-1 \\ -8 & c-6 \end{bmatrix}$ is not invertible.

b) Since $\det \begin{bmatrix} c & c-1 \\ -8 & c-6 \end{bmatrix} = c(c-6)+8(c-1)$

$= c^2-6c+8c-8$

$= c^2+2c-8$

$= (c+4)(c-2)$,

and $\begin{bmatrix} c & c-1 \\ -8 & c-6 \end{bmatrix}$ is not invertible when $\det \begin{bmatrix} c & c-1 \\ -8 & c-6 \end{bmatrix} = 0$,

we can conclude that $\begin{bmatrix} c & c-1 \\ -8 & c-6 \end{bmatrix}$ is not invertible when

$c = -4$ or $c = 2$.

3. (a) (2 points) Please give a definition of a subspace.

   (b) (8 points) Use the definition of a subspace to prove that

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \mid a + 5c = 0 \text{ and } 4b - 3d = 0 \right\}$$

is a subspace of $\mathbb{R}^4$. **You will not receive any points if you use other methods.**

b) Set $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a+5c=0 \text{ and } 4b-3d=0 \right\}$.

1) We will show that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in U$.

Since $0+5\cdot 0=0$ and $4(0)-3(0)=0$, we can conclude that

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in U$.

2) We will show that U is closed under vector addition.

Let $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \in U$. Hence, $a+5c=0$ ①, $4b-3d=0$ ②

and $e+5g=0$ ③, $4f-3h=0$ ④.

We want to show that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \in U$. It is enough to show that

$(a+e)+5(c+g)=0$ and $4(b+f)-3(d+h)=0$.

By ① & ③, we have $(a+e)+5(c+g)=(a+5c)+(e+5g)=0$ and by ② & ④, we have

$4(b+f)-3(d+h)=4b-3d+4f-3h=0$.

Hence, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \in U$ and U is closed under vector addition.

3) We will show that U is [illegible] under scalar multi.

Let $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U$ and let [illegible] be a scalar.

Since $a+5c=0$ and [illegible] $=0$, these imply that

$(\alpha a)+5(\alpha c)=\alpha(a+5$ [illegible] and

$4(\alpha b)-3(\alpha d)=\alpha(4b-$ [illegible] 0. Furthermore, $\alpha\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U$.

So, U is closed under [illegible] multi.

By 1) - 3), U is a subspace of $\mathbb{R}^4$.

4. (a) (2 points) Please give a definition of a null space of a matrix.

(b) (8 points) Find a basis and the dimension of the null space of a matrix $\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 6 & 3 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix}$.

b) Set $A = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 6 & 3 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix}$

$$\text{Null}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Since $\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 6 & 3 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, we can conclude

that $x_1 = -x_3 + x_5$, $x_2 = x_3$ and $x_4 = x_5$.

$$\text{Hence Null}(A) = \left\{ \begin{bmatrix} -x_3 + x_5 \\ x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} \middle| x_3, x_5 \text{ are scalars} \right\}$$

$$= \left\{ x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \middle| x_3, x_5 \text{ are scalars} \right\}.$$

A basis of $\text{Null}(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\dim \text{Null}(A) = 2$.

5. (10 points) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix} 1 \\ -2 \\ 5 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 1 \\ -4 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 8 \\ -3 \\ -5 \end{bmatrix}$.

(a) Find a basis and the dimension of W.

(b) Extend the basis of W to a basis of the whole space $\mathbb{R}^4$.

a) Since $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we can conclude that

a basis of W is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -4 \end{bmatrix} \right\}$ and $\dim W = 2$.

b) Since $\begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we can conclude that

$\left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is lin. indep and, in fact, it is

a basis of $\mathbb{R}^4$.