Solution for Chapter 1: Problems

1. Determine if the following statements are true or false.

   (a) If B is a 3×4 matrix, then its columns are 1×3 vectors.

   False. Its columns are 3×1 vectors.

   (b) Any scalar multiple of a vector $\vec{v}$ in $\mathbb{R}^n$ is a linear combination of $\vec{v}$.

   True

   (c) If a vector $\vec{v}$ lies in the span of a finite subset $\mathcal{S}$ of $\mathbb{R}^n$ then $\vec{v}$ is a linear combination of the vectors in $\mathcal{S}$.

   True

   (d) The matrix-vector product of an $m \times n$ matrix A and a vector in $\mathbb{R}^n$ is a linear combination of the columns of A.

   True

   (e) The rank of the coefficient matrix of a consistent system of linear equations is equal to the number of basic variables in the general solution to the system.

   True

   (f) The nullity of the coefficient matrix of a consistent system of linear equations is equal to the number of free variables in the general solution to the system.

   True

   (g) Every matrix can be transformed into one and only one matrix in reduced row echelon form by means of a sequence of elementary row operations.

   True

   (h) If the last row of the reduced row echelon form of an augmented matrix of a system of linear equations has only one nonzero entry, then the system is inconsistent.

   False. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$ **is a reduced row echelon form such that the last row has only one nonzero entry and the system is consistent.**

   (i) If the last row of the reduced row echelon form of an augmented matrix of a system of linear equations has only zero entries, then the system has infinitely many solutions.

   False. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **is a reduced row echelon form such that the last row has only zero entries and the system has a unique solution.**

   (j) The zero vector of $\mathbb{R}^n$ lies in the span of any finite subset of $\mathbb{R}^n$.

   True

(k) If the rank of an $m \times n$ matrix A is m then the rows of A are linearly independent.

True

(l) The columns of an $m \times n$ matrix A span $\mathbb{R}^m$ if and only if the rank of A is m.

True

(m) If the columns of an $m \times n$ matrix are linearly dependent then the rank of the matrix is less than m.

False. $\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$ **is a reduced row echelon form of a matrix such that its columns are linearly dependent and the rank of the given matrix(=3) equals the number of rows of the matrix.**

(n) If $\mathcal{S}$ is a linearly independent subset of $\mathbb{R}^n$ and $\vec{v}$ is a vector in $\mathbb{R}^n$ such that $\mathcal{S} \cup \{v\}$ is linearly dependent, then $\vec{v}$ is in the span of $\mathcal{S}$.

True

(o) A subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.

True

(p) A subset of $\mathbb{R}^n$ containing fewer than n vectors must be linearly independent.

False. A set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ **is a subset of** $\mathbb{R}^3$ **and this set is linearly dependent.**

(q) A linearly dependent subset of $\mathbb{R}^n$ must contain more than n vectors.

False. A set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ **is a subset of** $\mathbb{R}^3$ **and this set is linearly dependent.**

2. Let A be an $m \times n$ matrix, let $\vec{b}$ be a vector in $\mathbb{R}^m$, and suppose that $\vec{v}$ is a solution of $A\vec{x} = \vec{b}$.

(a) Prove that if $\vec{w}$ is a solution of $A\vec{x} = \vec{0}$ then $\vec{v} + \vec{w}$ is a solution of $A\vec{x} = \vec{b}$.

Proof. Since $\vec{w}$ is a solution of $A\vec{x} = \vec{0}$ and $\vec{v}$ is a solution of $A\vec{x} = \vec{b}$, we then have that $A\vec{w} = \vec{0}$ and $A\vec{v} = \vec{b}$. These imply that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{b} + \vec{0} = \vec{b}$. Hence $\vec{v} + \vec{w}$ is a solution of $A\vec{x} = \vec{b}$.

□

(b) Prove that for any solution $\vec{u}$ to $A\vec{x} = \vec{b}$, there is a solution $\vec{w}$ to $A\vec{x} = \vec{0}$ such that $\vec{u} = \vec{v} + \vec{w}$.

Proof. Let $\vec{u}$ be a solution of $A\vec{x} = \vec{b}$. Set $\vec{w} = \vec{u} - \vec{v}$. Since $A\vec{w} = A(\vec{u} - \vec{v}) = A\vec{u} - A\vec{v} = \vec{b} - \vec{b} = \vec{0}$, it follows that $\vec{w}$ is a solution of $A\vec{x} = \vec{0}$ and $\vec{u} = \vec{v} + \vec{w}$. □