<u>Magic Squares</u>

Recall that 1) V_3 is the set of all magic squares of order 3 and W_3 is the subset of V_3 consisting of magic squares with sum equal to 0.

2) Let C_3 be the 3×3 matrix all of whose entries are equal to $1/3$.

Let $A \in V_3$ and A has sum r. Then there is a unique magic square B in W_3 s.t $A = B + rC_3$. (*)

i) Find W_3.

Let $W = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \in W_3$.

Then
$$x_1 + x_2 + x_3 = 0$$
$$x_4 + x_5 + x_6 = 0$$
$$x_7 + x_8 + x_9 = 0$$
$$x_1 + x_4 + x_7 = 0$$
$$x_2 + x_5 + x_8 = 0$$
$$x_3 + x_6 + x_9 = 0$$
$$x_1 + x_5 + x_9 = 0$$
$$x_3 + x_5 + x_7 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $x_1 = -x_9$

$x_2 = -x_8$

$x_3 = x_8 + x_9$

$x_4 = x_8 + 2x_9$

$x_5 = 0$

$x_6 = -x_8 - 2x_9$

$x_7 = -x_8 - x_9$.

So $W_3 = \left\{ \begin{bmatrix} -t & -s & s+t \\ s+2t & 0 & -s-2t \\ -s-t & s & t \end{bmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} -t & 0 & t \\ 2t & 0 & -2t \\ -t & 0 & t \end{bmatrix} + \begin{bmatrix} 0 & -s & s \\ s & 0 & -s \\ -s & s & 0 \end{bmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

$= \left\{ t \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

ii) Find V_3

By (*), we can conclude that

$$V_3 = \left\{ t\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix} + s\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} + r\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \,\middle|\, t, s, r \in \mathbb{R} \right\}$$