MAT 347 Spring 2010: Quiz 4
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Name: Solution

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. Show that $(\mathbb{R} \times \mathbb{R}, d)$ is a metric space, where

$$d((x,y),(x',y')) = \begin{cases} |y| + |y'| + |x - x'| \text{ if } x \neq x' \\ |y - y'| \text{ if } x = x'. \end{cases}$$

2. Show that the subset of E^2 given by $\{(x_1, x_2) \in E^2 \; : \; x_1 > x_2\}$ is open.

1) i) Let $(x,y),(x',y') \in \mathbb{R}\times\mathbb{R}$. Since $|y| \geq 0$, $|y'| \geq 0$, $|x-x'| \geq 0$ and $|y-y'| \geq 0$, we can conclude that $d((x,y),(x',y')) \geq 0$.

ii) We will show that $d((x,y),(x',y')) = 0$ iff $(x,y) = (x',y')$

$(\leftarrow)$ If $(x,y) = (x',y')$ then $d((x,y),(x',y')) = |y-y'| = 0$.

$(\rightarrow)$ Assume that $d((x,y),(x',y')) = 0$.

If $x \neq x'$ then $|x-x'| > 0$ and $d((x,y),(x',y')) > 0$. This is impossible since $d((x,y),(x',y')) = 0$. Hence $x = x'$. $(*)$

Since $d((x,y),(x',y')) = 0$ and $d((x,y),(x',y')) = |y-y'|$, it follows that $|y-y'| = 0$ and $y = y'$. $(**)$. Consequently $(x,y) = (x',y')$

iii) Let $(x,y),(x',y') \in \mathbb{R}\times\mathbb{R}$. Clearly, $|y| + |y'| + |x-x'| = |y'| + |y| + |x'-x|$ and $|y-y'| = |y'-y|$. Therefore, $d((x,y),(x',y')) = d((x',y'),(x,y))$.

iv) Let $(a,b),(c,d),(e,f) \in \mathbb{R}\times\mathbb{R}$.

We will show that $d((a,b),(e,f)) \leq d((a,b),(c,d)) + d((c,d),(e,f))$

<u>case 1</u> $a = e$

<u>case 1.1</u> $c = a$

$d((a,b),(e,f)) = |b-f| = |b-d+d-f| \le |b-d| + |d-f| = d((a,b),(c,d)) + d((c,d),(e,f))$

<u>case 1.2</u> $c \ne a$

$d((a,b),(e,f)) = |b-f| \le |b| + |d| + |a-c| + |d| + |f| + |c-e| = d((a,b),(c,d))$
$+ d((c,d),(e,f))$.

<u>case 2</u> $a \ne e$

<u>case 2.1</u> $C = a$

$$\begin{aligned} d((a,b),(e,f)) = |b| + |f| + |a-e| &= |b-d+d| + |f| + |c-e| \\ &\le |b-d| + |d| + |f| + |c-e| \\ &= d((a,b),(c,d)) + d((c,d),(e,f)) \end{aligned}$$

<u>case 2.2</u> $c \ne a$

<u>case 2.2.1</u> $c = e$

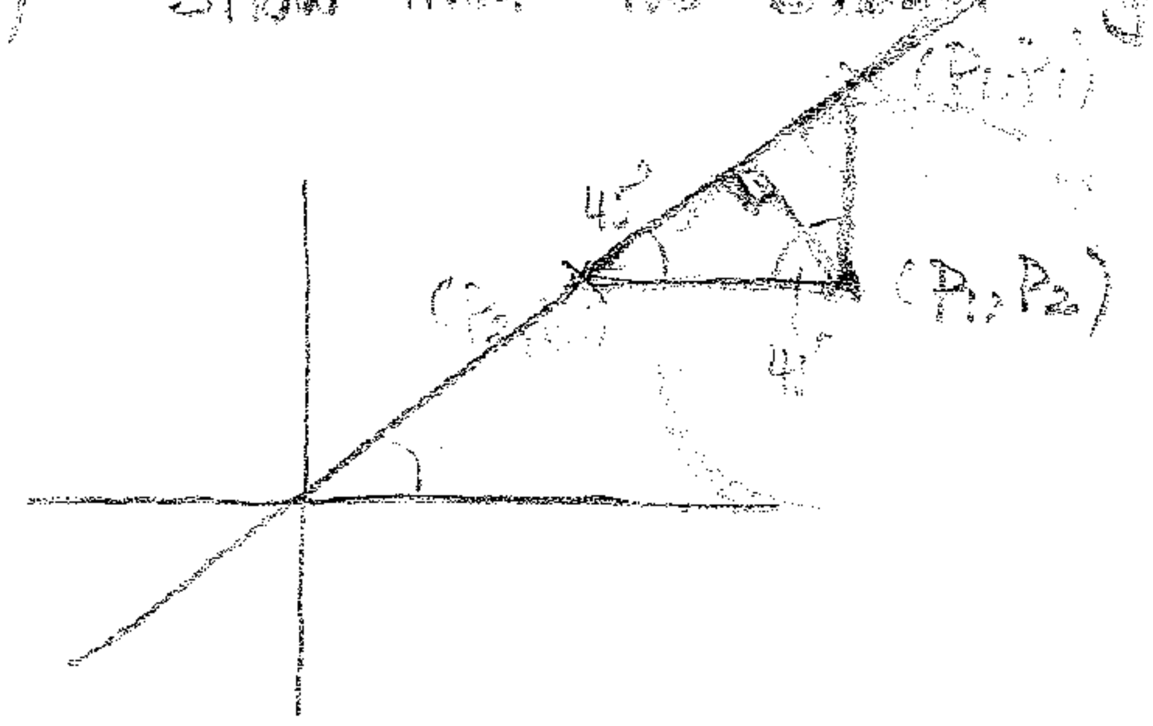
$$\begin{aligned} d((a,b),(e,f)) = |b| + |f| + |a-e| &= |b| + |d-d+f| + |a-c| \\ &\le |b| + |d| + |a-c| + |d-f| \\ &= d((a,b),(c,d)) + d((c,d),(e,f)) \end{aligned}$$

<u>case 2.2.2</u> $c \ne e$

$$\begin{aligned} d((a,b),(e,f)) = |b| + |f| + |a-e| &\le |b| + |d| + |a-c| + |d| + |f| + |c-e| \\ &= d((a,b),(c,d)) + d((c,d),(e,f)) \end{aligned}$$

By i) – iv), we can conclude that $(\mathbb{R}\times\mathbb{R}, d)$ is a metric space.

2) Show that the subset of $\mathbb{R}^2$ given by $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\}$ is open



Let $S = \{(x_1, x_2) \in E^2 \mid x_1 > x_2\}$

Let $p = (p_1, p_2) \in S$. The shortest distance from p to the line $y = x$ is

$$\frac{p_1 - p_2}{\sqrt{2}}.$$

Set $r = \dfrac{p_1 - p_2}{\sqrt{2}}$. We will show that $N(p, r) \subset S$.

Let $q = (q_1, q_2) \in N(p, r)$. If $q_1 \leq q_2$ then $q = (q_1, q_2)$ will be in the region that is located on the left side of the line $x_1 = x_2$. Since the shortest distance from p to the area $x_1 \leq x_2$ is r, it implies that $d(p, q) \geq r$. This is impossible since $q \in N(p, r)$. Hence $q_1 > q_2$ and $q = (q_1, q_2) \in S$.

Consequently, $N(p, r) \subset S$ and S is open.