Chapter IV problem 8

Let $U = \{x \in \mathbb{R} \mid x > a\}$, for some positive real number a and let f be a real-valued function on U. Define $\lim_{x\to\infty} f(x) = \lim_{y\to 0} g(y)$ where $g : (0, 1/a) \to \mathbb{R}$ is given by $g(y) = f(1/y)$, if this latter limit exists.

Prove that $\lim_{x\to\infty} f(x)$ exists $\longleftrightarrow$ $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t if $x, y \in \mathbb{R}$ and $x, y > N$ then $|f(x) - f(y)| < \varepsilon$.

Proof ($\rightarrow$) Assume $\lim_{x\to\infty} f(x)$ exists. Then $\lim_{y\to 0} g(y)$ exists.

Set $A = \lim_{y\to 0} g(y)$

Let $\varepsilon > 0$. $\exists\, \delta > 0$ s.t if $b \in (0, 1/a)$ s.t $d(b, 0) < \delta$ then $d'(g(b), A) < \varepsilon/2$.

By AP, $\exists N \in \mathbb{N}$ s.t $\frac{1}{N} < \delta$. Hence for all $c, d \in (0, 1/a)$ s.t $c, d < \frac{1}{N}$, we then have that

$|g(c) - g(d)| = |g(c) - A + A - g(d)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $(*)$

Let $x, y \in \mathbb{R}$ s.t $x, y > N$. ~~We H~~ Hence, $\frac{1}{x} < \frac{1}{N}$ and $\frac{1}{y} < \frac{1}{N}$. By $(*)$, we have $|f(x) - f(y)| = |g(\frac{1}{x}) - g(\frac{1}{y})| < \varepsilon$.

($\leftarrow$) Assume that $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t if $x, y \in \mathbb{R}$ and $x, y > N$ then $|f(x) - f(y)| < \varepsilon$.

Suppose $\lim_{x\to\infty} f(x)$ does not exist. Hence, $\lim_{y\to 0} g(y)$ does not exist.

So, $\exists$ a seq $(y_n)_{n\in\mathbb{N}}$ in $(0, \frac{1}{a})$ s.t $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} g(y_n)$ does not exist.

Let $\varepsilon > 0$. By assumption $\exists N \in \mathbb{N}$ s.t if $x, y \in \mathbb{R}$, and $x, y > N$ then $|f(x) - f(y)| < \varepsilon$. Since $\lim_{n\to\infty} y_n = 0$, it follows that $\exists N_1 \in \mathbb{N}$ s.t $|y_n| < \frac{1}{N}$ for all $n > N_1$. Therefore $N < \frac{1}{y_n}$ for all $n > N_1$ and ~~$|f(\frac{1}{y}$~~ $|g(y_n) - g(y_m)| = |f(\frac{1}{y_n}) - f(\frac{1}{y_m})| < \varepsilon$ for all $m, n > N_1$. Therefore $(g(y_n))_{n\in\mathbb{N}}$ is a Cauchy seq in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, ~~it~~ we can conclude that $(g(y_n))_{n\in\mathbb{N}}$ is a convergent seq. This is a contradiction.

Hence, $\lim_{x\to\infty} f(x)$ exists.