Chapter IV problem 2

Let E, E' be metric spaces, $f: E \to E'$ a continuous function.
Show that if S is a closed subset of E' then $f^{-1}(S)$ is a closed subset of E. Derive from this the results that if f is a continuous real-valued function on E then the sets $\{p \in E \mid f(p) \le 0\}$, $\{p \in E \mid f(p) \ge 0\}$, $\{p \in E \mid f(p) = 0\}$ are closed.

Proof Assume that S is a closed subset of E'. Then $E' \setminus S$ is open. So, $f^{-1}(E' \setminus S)$ is open since f is cont.

Claim $f^{-1}(E' \setminus S) = E \setminus f^{-1}(S)$

($\subseteq$) Let $x \in f^{-1}(E' \setminus S)$. Then $f(x) \in E' \setminus S$. ~~Hence~~
This implies that $f(x) \notin S$ and $x \notin f^{-1}(S)$.
Therefore $x \in E \setminus f^{-1}(S)$.

($\supseteq$) Let $y \in E \setminus f^{-1}(S)$. Hence $y \notin f^{-1}(S)$ and $f(y) \notin S$, ($\to f(y) \in E' \setminus S$)
Consequently, $y \in f^{-1}(E' \setminus S)$

Since $f^{-1}(E' \setminus S)$ is open and $f^{-1}(E' \setminus S) = E \setminus f^{-1}(S)$, we can conclude that $E \setminus f^{-1}(S)$ is open and $f^{-1}(S)$ is closed.

Since $(-\infty, 0]$, $[0, \infty)$, and $\{0\}$ are closed, we can conclude that $\{p \in E \mid f(p) \le 0\}$, $\{p \in E \mid f(p) \ge 0\}$, $\{p \in E \mid f(p) = 0\}$ are closed

Chapter IV problem 3

Let E, E' be metric spaces, $f: E \to E'$ a function, and suppose that S_1, S_2 are closed subsets of E s.t $E = S_1 \cup S_2$.

Show that if $f|_{S_1}$ and $f|_{S_2}$ are continuous then f is cont.

Proof. Let $x \in S_1 \cup S_2$.

Case 1 $x \in \operatorname{int} S_1$ or $x \in \operatorname{int} S_2$.

Let us assume that $x \in \operatorname{int} S_1$. Then $\exists \delta_1 > 0$ s.t $N(x, \delta_1) \subset S_1$. Let $\varepsilon > 0$. Since $f|_{S_1}$ is cont at x_1, it implies that $\exists \delta_2 > 0$ s.t if $q \in S_1$ and $d(x,q) < \delta_1$ then $d'(f(x), f(q)) < \varepsilon$.

Now, we will show that f is cont at x.

Let $\varepsilon > 0$. Set $\delta = \min\{\delta_1, \delta_2\}$. ~~For~~ $q \in E$ s.t $d(q,x) < \delta$, ~~Then~~ we have that $q \in S_1$ and $d'(f(x), f(q)) < \varepsilon$.

So, f is cont at x.

Similarly, if $x \in \operatorname{int} S_2$, f is also cont at x.

Case 2 $x \notin \operatorname{int} S_1$ and $x \notin \operatorname{int} S_2$.

Recall $E = \operatorname{int} S_1 \cup \operatorname{int} \mathcal{C} S_1 \cup \operatorname{Bd} S_1$, and $\mathcal{C} S_1 = S_2$.

Since $x \notin \operatorname{int} S_1$ and $x \notin \operatorname{int} S_2$, it follows that $x \in \operatorname{Bd} S_1$.

Since S_1 and S_2 are closed, we can conclude that

$\operatorname{Bd} S_1 = \overline{S_1} \cap \overline{\mathcal{C} S_1} = \overline{S_1} \cap \overline{S_2} = S_1 \cap S_2$.

Let $\varepsilon > 0$. Since $f|_{S_1}$ and $f|_{S_2}$ are cont at x, it follows that $\exists\, \delta_1, \cancel{\delta_2} > 0$ s.t

if $q \in S_1$ and $d(q, x) < \delta_1$ then $d'(f(x), f(q)) < \varepsilon$

and $\exists\, \delta_2 > 0$ s.t

if $p \in S_2$ and $d(p, x) < \delta_2$ then $d'(f(x), f(p)) < \varepsilon$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in E$ s.t $d(x, y) < \delta$.

Since $E = S_1 \cup S_2$, we then have that $y \in S_1$ or $y \in S_2$.

In either case, we have that $d'(f(x), f(y)) < \varepsilon$.

Hence f is cont at x.

Chapter IV problem 4

Let U, V be (open or closed) intervals in $\mathbb{R}$, and let $f: U \to V$ be a fcn which is strictly increasing and onto.

Prove f and f^{-1} are cont.

<u>Proof</u> Since f is strictly increasing, it implies that f is 1-1.

Since f is 1-1, onto, it follows that f^{-1} exists.

Assume that $U = [A, B]$.

Let $x \in U$. Let $\varepsilon > 0$.

<u>case 1</u> $N(f(x), \varepsilon) \supseteq V$. Then $N(f(x), \varepsilon) \cap V = V$

and $f^{-1}(N(f(x), \varepsilon) \cap V) = f^{-1}(V) = U$,

f is onto

Since U is open as a metric space, it implies that

$\exists\, \delta > 0$ s.t $N(x, \delta) \cap U \subset U$. Clearly,

if $y \in N(x, \delta) \cap U$ then $f(y) \in V = N(f(x), \varepsilon) \cap V$.

<u>case 2</u> $N(f(x), \varepsilon) \subset V$. Then $N(f(x), \varepsilon) \cap V = N(f(x), \varepsilon)$

$= (f(x) - \varepsilon, f(x) + \varepsilon)$. Since f is onto, it follows that

$\exists\, c, d \in U$ s.t $f(c) = f(x) - \varepsilon$, $f(d) = f(x) + \varepsilon$.

Set $\delta = \min\{x - c, d - x\}$.

Let $y \in N(x, \delta)$. Then $c < y < d$. ~~Hen~~ and

$f(c) < f(y) < f(d)$. So, $f(y) \in (f(c), f(d)) = (f(x) - \varepsilon, f(x) + \varepsilon)$.

By case 1 and case 2 we can conclude that f is cont at x.

In fact f is cont on U. Similarly, f^{-1} is cont on V.

Chapter IV problem 6

Let E be a metric space, S a subset of E and let $f: E \to \mathbb{R}$ be the function which takes the value 1 at each point of S and 0 at each point of $\complement S$.

Prove that the set of points of E at which f is not cont is precisely the boundary of S.

Proof Recall that $E = \text{int } S \cup \text{int } \complement S \cup \text{Bd } S$ and $\text{int } S \cap \text{int } \complement S = \phi$, $\text{int } S \cap \text{Bd } S = \phi$ and $\text{int } \complement S \cap \text{Bd } S = \phi$.

Since $f|_{\text{int } S}$ and $f|_{\text{int } \complement S}$ are constant fcns, we can conclude that ~~$f|_{\Omega}$~~ f is cont on $\text{int } S \cup \text{int } \complement S$.

Let $p \in \text{Bd } S = \overline{S} \cap \overline{\complement S}$. Assume that f is cont at p.

~~Since $\overline{S}$ is closed, and $p \in \overline{S}$,~~

Since $p \in \overline{S}$ and $\overline{S}$ is the set of all limits of sequences of points of S the converge in E, it follows that $\exists$ a seq. $(p_n)_{n \in \mathbb{N}}$ in S s.t $\lim_{n \to \infty} p_n = p$. Since f is cont at p, it implies that

$$f(p) = \lim_{n \to \infty} f(p_n) = 1. \quad (*)$$

Similarly, since $p \in \overline{\complement S}$, it implies that $\exists$ a seq $(q_n)_{n \in \mathbb{N}}$ in $\complement S$ s.t $\lim_{n \to \infty} q_n = p$. Since f is cont at p, we then have that

$$f(p) = \lim_{n \to \infty} f(q_n) = 0 \quad (**)$$. This is a contradiction.

Hence f is not cont at p. In fact, f is not cont on $\text{Bd}(S)$.