Chapter III problem 9

Prove that $\lim_{n\to\infty} p_n = p$ in a given metric space iff the sequence $p_1, p, p_2, p, p_3, p, \cdots$

is convergent

Proof ($\rightarrow$) Set $(q_n)_{n\in\mathbb{Z}}$ = sequence $p_1, p, p_2, p, p_3, \cdots$

Goal We will show that $\lim_{n\to\infty} q_n = p$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} p_n = p$, it implies that $\exists N \in \mathbb{N}$ s.t

$d(p_n, p) < \varepsilon$ for all $n > N$.

Since $(q_n)_{n\in\mathbb{N}}$ = seq $p_1, p, p_2, p, p_3, \cdots$, it follows that

$\exists M \in \mathbb{N}$ s.t $q_M = p_{N+1}$. Hence for $m > M$, we have

that q_m ~~is at~~ equals either p or p_j where $j > N+1$.

In either case
~~Furthermore~~, we have that $d(q_m, p) < \varepsilon$ for all $m > M$.

Hence, $\lim_{n\to\infty} q_n = p$.

($\leftarrow$) Assume that $(q_n)_{n\in\mathbb{N}}$ (= seq $p_1, p, p_2, p, p_3, \cdots$) is convergent.

Since $(q_{2n})_{n\in\mathbb{N}} = (p)_{n\in\mathbb{N}}$ is a subsequence of $(q_n)_{n\in\mathbb{N}}$

and $\lim_{n\to\infty} q_{2n} = p$, we can conclude that $\lim_{n\to\infty} q_n = p$.

Since $(p_n)_{n\in\mathbb{N}} = (q_{2n-1})_{n\in\mathbb{N}}$ is a subseq of $(q_n)_{n\in\mathbb{N}}$, we

can conclude that $\lim_{n\to\infty} p_n = p$.

Chapter III Problem 10

Prove that if $\lim_{n\to\infty} p_n = p$ in a given metric space, then the set of points $\{p, p_1, p_2, p_3, \dots\}$ is closed.

Proof Claim: If the points of a convergent sequence of points in a metric space are reordered, then the new sequence converges to the same point.

Proof of Claim. Assume that $(p_n)_{n\in\mathbb{N}}$ is a conv seq and $\lim_{n\to\infty} p_n = p$. Let $(q_m)_{m\in\mathbb{N}}$ be a new seq ~~of~~ obtained from the seq $(p_n)_{n\in\mathbb{N}}$ by reordering.

We will show that $\lim_{m\to\infty} q_m = p$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} p_n = p$, it implies that $\exists N \in \mathbb{N}$ s.t $d(p_n, p) < \varepsilon$ for all $n > N$.

Case 1 $\{q_m \mid m\in\mathbb{N}\} (= \{p_n \mid n\in\mathbb{N}\})$ is a finite set.

For this case there is constant term q s.t $q_m = q$ for infinitely many m. Since $\{q_m \mid m\in\mathbb{N}\} = \{p_n \mid n\in\mathbb{N}\}$ and $\lim_{n\to\infty} p_n = p$, it follows that $q = p$, and $\lim_{m\to\infty} q_m = p$.

Case 2 $\{q_m \mid m\in\mathbb{N}\}$ is an infinite set.

For this case there is an $M\in\mathbb{N}$ s.t for all $m > M$ we have $q_m = p_n$ where $n > N$.

Hence $d(q_m, p) < \varepsilon$ for all $m > M$ and $\lim_{n\to\infty} q_m = p$.

Proof of the main problem. Let $(r_n)_{n\in\mathbb{N}}$ be a seq in S. Then $(r_n)_{n\in\mathbb{N}}$ is either a seq that is obtained from $(p_n)_{n\in\mathbb{N}}$ by reordering or a subseq of $(p_n)_{n\in\mathbb{N}}$ or a subseq of a reordered seq. Hence $(r_n)_{n\in\mathbb{N}}$ is a convergent seq, and

$\lim_{n\to\infty} r_n = p \in S$.

Hence, S is closed.

Chapter III Problem 11

Show that if $(a_n)_{n\in\mathbb{N}}$ is a seq of real numbers that converges to a then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a.$$

Proof Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, it follows that $\exists N \in \mathbb{N}$ s.t $|a_n - a| < \varepsilon/2$ for all $n > N$. Set $B = \sum_{i=1}^{N} (a_i - a)$. Since $\lim_{n\to\infty} \frac{B}{n} = 0$, it implies that $\exists M \in \mathbb{N}$ s.t $\left|\frac{B}{n} - 0\right| < \varepsilon/2$ for all $n > M$.

Set $\tilde{N} = \max\{M, N\}$. For $n > \tilde{N}$, we have

$$\left|\frac{\sum_{i=1}^{n} a_i}{n} - a\right| = \left|\frac{\sum_{i=1}^{N}(a_i - a)}{n} + \sum_{i=N+1}^{n} \frac{a_i - a}{n}\right|$$

$$\leq \underbrace{\left|\sum_{i=1}^{N} \frac{(a_i - a)}{n}\right|}_{= |B/n|} + \sum_{i=N+1}^{n} \frac{|a_i - a|}{n}$$

$$< \varepsilon/2 + \frac{1}{n}(n - (N+1))\,\varepsilon/2$$

$$= \varepsilon/2 + \left(1 - \frac{(N+1)}{n}\right)\varepsilon/2$$

$\to \leq 1$ and greater than or equal to zero.

$$< \varepsilon$$

Hence $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a.$