Chapter III problem 26

Find all cluster points of the subset of $\mathbb{R}$ given by $\{\frac{1}{n}+\frac{1}{m} \mid n,m \in \mathbb{N}\}$.

Proof Set $S = \{\frac{1}{n}+\frac{1}{m} \mid n,m \in \mathbb{N}\}$.

Claim For $n\in\mathbb{N}$, $\frac{1}{n}$ is a cluster point.

Let $\varepsilon>0$. By AP, $\exists M\in\mathbb{N}$ s.t $\frac{1}{M}<\varepsilon$. Furthermore $\frac{1}{m}<\varepsilon$ $\forall m>M$.

Since $d(\frac{1}{n}, \frac{1}{n}+\frac{1}{m}) = \frac{1}{m}<\varepsilon$ for all $m>M$, we can conclude that

$\frac{1}{n}+\frac{1}{m}\in N(\frac{1}{n},\varepsilon)$ for all $m>M$. So, $\frac{1}{n}$ is a cluster point.

Claim 0 is a cluster point.

Let $\varepsilon>0$. By AP, $\exists M\in\mathbb{N}$ s.t $\frac{1}{M}<\varepsilon/2$. In fact $\frac{1}{m}<\varepsilon/2$ $\forall m>M$.

Since $d(0, \frac{1}{M+1}+\frac{1}{m}) = \frac{1}{M+1}+\frac{1}{m} < \varepsilon/2+\varepsilon/2$ for all $m>M$,

we can conclude that $\frac{1}{M+1}+\frac{1}{m}\in N(0,\varepsilon)$ for all $m>M$,

and 0 is a cluster point.

Claim 0 and $\frac{1}{n}$ for all $n\in\mathbb{N}$ are the only cluster points of S.

~~By Th~~ Let p be a cluster point of S. By def^n of cluster point, we know that p is a limit of some seq in S, says $(q_i)_{i\in\mathbb{N}}$.

However, subsequences of $(q_i)_{i\in\mathbb{N}}$ will converge to either 0 or $\frac{1}{n}$ for some $n\in\mathbb{N}$. It follows that p is either 0 or $\frac{1}{n}$ for some $n\in\mathbb{N}$,

Chapter III problem 27

Let $S \neq \phi$ be a subset of $\mathbb{R}$ that is bounded from above but has no greatest elt. Prove that l.u.b S is a cluster point of S.

Proof Let $y = $ l.u.b. Since $y-1$ is not an upper bound, it implies that $\exists s_1 \in S$ s.t $y-1 < s_1 < y$. Since $y - \frac{1}{2}$ is not an u.b., and S has no greatest elt, it follows that $\exists s_2 \in S$ s.t $y - \frac{1}{2} < s_2 < y$ and $s_1 < s_2$. Since $y - \frac{1}{3}$ is not an u.b, and S has no greatest elt, we then have that $\exists s_3 \in S$ s.t $y - \frac{1}{3} < s_3 < y$ and $s_1 < s_2 < s_3$. Continue in this way we can construct a seq $(s_n)_{n \in \mathbb{N}}$ s.t $y - \frac{1}{n} < s_n < y$ and $s_1 < s_2 < \dots < s_n < \dots$

Let $\mathcal{E} > 0$. By A.P, $\exists M \in \mathbb{N}$ s.t $\frac{1}{M} < \mathcal{E}$. Hence $y - \frac{1}{m} > y - \mathcal{E}$ for all $m > M$ and $s_m \in N(y, \mathcal{E})$ for all $m > M$.

We then have that y is a cluster point

Chapter III problem 28

Prove that a subset S of a metric space is closed $\Longleftrightarrow$ it contains all its cluster points.

Proof ($\rightarrow$) Assume that S is closed.

Let p be a cluster point of S. Then p is the limit of a seq in S. Since S is closed, it follows that $p \in S$.

($\leftarrow$) Assume that S contains all its cluster points.

So, S contains all limits of sequences of points of S that converge in E. By 16 b), we can conclude that $S = \overline{S}$.

Chapter III problem 30

Give an example of each of the following:

a) an infinite subset of $\mathbb{R}$ with no cluster point.

$\mathbb{N}$.

b) a complete metric space that is bounded but not compact.

A metric space $(\mathbb{N}, d)$ is a complete metric space that is bounded but not compact. Here, the distance function d is defined in the following way:

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases}$$

Claim $(\mathbb{N}, d)$ is complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy seq of points of $(\mathbb{N}, d)$.

Let $\varepsilon > 0$ and $\varepsilon < 1$. Then $\exists N \in \mathbb{N}$ s.t $d(x_n, x_m) < \varepsilon < 1$ for all $n, m > N$. Hence $d(x_n, x_m) = 0$ and $x_n = x_{N+1}$ for all $n > N$.

Clearly, $(x_n)_{n\in\mathbb{N}}$ is a convergent seq. Furthermore, $(\mathbb{N}, d)$ is complete.

Claim $(\mathbb{N}, d)$ is bounded

$$(\mathbb{N}, d) \subset \underbrace{B(\overset{\text{center at } 1}{1}, \overset{\text{radius} = 2}{2})}_{\text{closed ball}}$$

Claim $(\mathbb{N}, d)$ is not compact

$$\mathbb{N} \subset \bigcup_{i \in \mathbb{N}} N(i, 1/2) \quad \text{and} \quad \mathbb{N} \not\subset N(i_1, 1/2) \cup N(i_2, 1/2) \ldots \cup N(i_k, 1/2)$$

for any $k \in \mathbb{N}$.

c) a metric space none of whose closed balls is complete.

Consider $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Claim that none of closed balls is complete.

Let $x \in S$ and $\varepsilon > 0$. We will denote the closed ball center at x and radius ε by $B(x, \varepsilon)$. ~~Assume that $B(x,\varepsilon)$ is complete~~.

~~Let $(q_m)_{m \in}$~~ Assume that $B(x, \varepsilon) \cap S$ is complete.

Let $(q_m)_{m \in \mathbb{N}}$ be a Cauchy seq in $B(x, \varepsilon) \cap S$ s.t $\{q_m \mid m \in \mathbb{N}\}$ is an infinite set.

Hence $(q_m)_{m \in \mathbb{N}}$ is a convergent seq and its limit is in $B(x, \varepsilon) \cap S$.

Let $(q_{m_k})_{k \in \mathbb{N}}$ be ~~a seq~~ a subseq of $(q_m)_{m \in \mathbb{N}}$ satisfies the ~~defined by~~ property that $q_{m_1} > q_{m_2} > q_{m_3} > \ldots > q_{m_k} > \ldots$. Note we can write $q_{m_k} = \frac{1}{m_k}$. Clearly $\lim_{k \to \infty} q_{m_k} = 0$. Hence $\lim_{m \to \infty} q_m = 0$ as well.

This contradicts with that assumption that $B(x, \varepsilon) \cap S$ is complete.

So, $B(x, \varepsilon) \cap S$ is not complete.

Chapter III problem 32.

Show that the union of a finite number of compact subsets of a metric space is compact.

Proof Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ be compact set.

Let $\{O_i\}_{i \in I}$ be a collection of open sets s.t $\bigcup_{j=1}^{k} \mathcal{C}_j \subset \bigcup_{i \in I} O_i$.

For each $j \in \{1, \dots, k\}$, since $\mathcal{C}_j$ is compact, it implies that

$$\cancel{\mathcal{C}_j \subset O_{i_{1,j}} \cup O_{i_{2,j}} \cup \dots \cup O_{i_{l_j}}}$$

$$\mathcal{C}_j \subset O_{j_1} \cup O_{j_2} \cup \dots \cup O_{j_l}.$$

Therefore $\bigcup_{j=1}^{k} \mathcal{C}_j \subset \bigcup_{j=1}^{k} (O_{j_1} \cup O_{j_2} \cup \dots \cup O_{j_l})$.

and $\bigcup_{j=1}^{k} \mathcal{C}_j$ is compact.