Chapter III problem 15

Let S be a subset of the metric space E. A point $p \in S$ is called an interior point of S if there is an open ball in E of center p which is contained in S. Prove that the set of interior points of S is an open subset of E (called the interior of S) that contains all other open subsets of E that are contained in S.

Proof We will denote the interior of S by $int(S)$.

1) We will show that $int(S)$ contains all other open subsets of E that are contained in S.

Proof Let O be an open subset of E that is contained in S.

Goal WTS $O \subset int(S)$.

Let $p \in O$. Since O is open, it implies that $\exists\, \varepsilon > 0$ s.t $N(p, \varepsilon) \subset O$. Since $O \subset S$, it follows that $N(p, \varepsilon) \subset S$.

Hence p is an interior point of S, and $p \in int(S)$.

Furthermore, we have that $O \subset Int(S)$.

2) We will show that $int(S)$ is an open set.

Proof Let $p \in int(S)$. Hence, $\exists\, \delta > 0$ s.t $N(p, \delta) \subset S$ (by def ^{n}).

By 1), we then have that $N(p, \delta) \subset int(S)$. Hence, $int(S)$ is open.

Chapter III problem 16

Let S be a subset of the metric space E. Define the closure of S, denoted $\bar{S}$, to be the intersection of all closed subsets of E that contain S. Show that

a) $\bar{S} \supset S$, and S is closed iff $\bar{S} = S$.

Proof 1) $\bar{S} \supset S$.

Let $\{B_i\}_{i\in I}$ be a collection of all closed subsets of E that contain S. Then $\bar{S} = \bigcap_{i\in I} B_i$. If $x \in S$ then $x \in B_i$ for all $i \in I$.

Hence $x \in \bar{S}$. Therefore, $S \subset \bar{S}$.

2) S is closed $\longleftrightarrow$ $\bar{S} = S$.

($\rightarrow$) If S is closed then $\bar{S} \subset S$ (by def^{n} of $\bar{S}$).

Since $\underbrace{\bar{S} \supset S}_{\text{by}(1)}$ and $\bar{S} \subset S$, it follows that $\bar{S} = S$.

($\leftarrow$) Assume $\bar{S} = S$. Since $\bar{S}$ is the intersection of all closed subsets of E that contain S, it implies that $\bar{S}$ is closed. Since $S = \bar{S}$, we can conclude that S is closed.

b) $\bar{S}$ is the set of all limits of sequences of points of S that converge in E.

Proof Set S' be the set of all limits of ~~seqs~~ sequences of points of S that converge in E.

Let $x \in S'$. If $x \notin \bar{S}$ then $x \in \mathcal{C}(\bar{S})$. Since $\bar{S}$ is closed, it implies that $\mathcal{C}(\bar{S})$ is open and $\exists\ \varepsilon > 0$ s.t. $N(x, \varepsilon) \subset \mathcal{C}(\bar{S})$. This is impossible since x is a limit of a seq in S. Hence $x \in \bar{S}$. So, $S' \subset \bar{S}$. ①

Next, we will show that S' is a closed set that contains S.

i) $S \subset S'$. <u>Proof</u> Let $x \in S$. ~~There~~ Since x is the limit of a seq $(x)_{n\in\mathbb{N}}$.
Hence $x \in S'$.

ii) S' is closed.

<u>Proof</u>. ~~Let y be a the limit of a seq $(p_n)_{n\in\mathbb{N}}$ in S'.~~

<u>Recall</u> that S' is closed $\longleftrightarrow$ whenever $(p_n)_{n\in\mathbb{N}}$ is a seq in S' that is convergent in E we have $\lim_{n\to\infty} p_n \in S'$.

Let $(p_n)_{n\in\mathbb{N}}$ be a convergent seq in E and $p_n \in S'$ for all $n \in \mathbb{N}$.

Set $p = \lim_{n\to\infty} p_n$. We will show that $p \in S'$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t $d(p, p_n) < \varepsilon/2$ for all $n > N$. $(*)$

Since $p_{N+1} \in S'$, it follows that p_{n+1} is the limit of seq of points in S, says $(q_m)_{m\in\mathbb{N}}$. Hence $\exists M \in \mathbb{N}$ s.t $d(p_{N+1}, q_m) < \varepsilon/2$ for all $m > M$ $(*)$

~~Claim~~ By $(*)$, $(**)$, we have that
$$d(p, q_m) \le d(p, p_{N+1}) + d(p_{N+1}, q_m) < \varepsilon/2 + \varepsilon/2 \text{ for all } m > M.$$

Hence p is also ~~a~~ the limit of $(q_m)_{m\in\mathbb{N}}$. ~~B~~ and $p \in S'$.

These imply that S' is closed.

Since S' is a closed set that contains S, we can conclude that $\overline{S} \subset S'$. ② By ① & ②, we then have that $\overline{S} = S'$.

c) a point $p \in E$ is in $\bar{S} \longleftrightarrow$ any ball in E of center p contains points of S, which is true $\longleftrightarrow$ p is not an interior point of $\complement S$.

Proof i) We will show that any ball in E of center p contains points of S $\longleftrightarrow$ p is not an interior point of $\complement S$.

Proof of i) $(\rightarrow)$ Assume that any ball in E of center p contains points of S

If p is an interior point of $\complement S$, it follows that $\exists \delta > 0$ s.t $N(p,\delta) \subset \complement S$. This is impossible since $N(p,\delta) \cap S \neq \phi$.

So, p is not an interior point of $\complement S$.

$(\leftarrow)$ Assume p is not an interior point of $\complement S$. So, For each $\varepsilon > 0$, Hence for all $\varepsilon > 0$, $N(p,\varepsilon) \not\subset \complement S$. there exists $s_\varepsilon \in S$ s.t $s_\varepsilon \in N(p,\varepsilon)$.

2) We will show that $p \in \bar{S} \leftrightarrow \forall \varepsilon > 0,\ N(p,\varepsilon) \cap S \neq \phi$.

Proof of 2) $(\rightarrow)$ Assume that $p \in \bar{S}$. Then p is a limit of a seq in S, says $(q_m)_{m \in \mathbb{N}}$. Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $d(p, q_m) < \varepsilon$ for all $m > N$. Since $q_m \in S$ for all $m \in \mathbb{N}$, and ~~d(p,q_n~~ $q_m \in N(p,\varepsilon)$ for all $m > N$, it follows that $N(p,\varepsilon) \cap S \neq \phi$.

$(\leftarrow)$ ~~If~~ Assume that ~~for~~ $N(p,\varepsilon) \cap S \neq \phi$ for all $\varepsilon > 0$. Hence p is the limit of a seq in S. So, $p \in S' = \bar{S}$. ◻

Chapter III Problem 17

Let S be a subset of the metric space E. The boundary of S is defined to be $\bar{S} \cap \overline{\mathcal{C}S}$. Show that

a) E is the disjoint union of the interior of S, the interior of $\mathcal{C}S$, and the boundary of S.

Proof. Let $x \in E$, and $x \notin \text{int } S$. We will show that $x \in \text{int}(\mathcal{C}S) \cap$ (the boundary of S). Since $x \notin \text{int } S$, it follows that $N(x, \varepsilon) \cap \mathcal{C}S \neq \phi$ for all $\varepsilon > 0$.

case 1 $\exists \varepsilon > 0$ s.t. $N(x, \varepsilon) \subset \mathcal{C}S$.

Then $x \in \text{int}(\mathcal{C}S)$.

case 2 $\forall \varepsilon > 0$, $N(x, \varepsilon) \not\subset \mathcal{C}S$. By 16 (C),

Hence $N(x, \varepsilon) \cap S \neq \phi$ for all $\varepsilon > 0$. (*) Similarly, since we can conclude that $x \in \bar{S}$. $N(x, \varepsilon) \cap \mathcal{C}S \neq \phi$ for all $\varepsilon > 0$, we then have that $x \in \overline{\mathcal{C}S}$. (**)

By (**), we conclude that $x \in \bar{S} \cap \overline{\mathcal{C}S}$ (= the boundary of S).

iii) Similarly, $\text{int}(\mathcal{C}S) \cap$ (the boundary of S) $= \phi$

So, $E = \text{int}(S) \cup \text{int}(\mathcal{C}S) \cup$ (the boundary of S).

Next, we will show that i) $\text{int}(S) \cap \text{int}(\mathcal{C}S) = \phi$, ii) $\text{int}(S) \cap$ (the boundary of S) $= \phi$, and iii) $\text{int}(\mathcal{C}S) \cap$ (the boundary of S) $= \phi$.

i) Since $\text{int } S \subset S$, $\text{int } \mathcal{C}S \subset \mathcal{C}S$ and $S \cap \mathcal{C}S = \phi$, we can conclude that $\text{int } S \cap \text{int } \mathcal{C}S = \phi$.

ii) If $\text{int}(S) \cap$ the boundary of $S \neq \phi$, then $\exists x \in E$ s.t. $x \in \text{int } S$ and $x \in \bar{S} \cap \overline{\mathcal{C}S}$. Since $x \in \overline{\mathcal{C}S}$, by (16 c), we can conclude that x is not an interior point of $\mathcal{C}(\mathcal{C}S) = S$. This is impossible. So, $\text{int } S \cap$ (the boundary of S) $= \phi$

b) S is closed if and only if S contains its boundary.

Proof ~~Rec~~ Set $Bd(S)$ = the boundary of S.

Since $E = \text{int } S \cup \text{int } \mathcal{C}S \cup Bd(S)$, it implies that (*)

$$S = ((\text{int } S) \cap S) \cup (\text{int } \mathcal{C}S \cap S) \cup ((Bd\ S) \cap S) \quad (*)$$

Since $\text{int } S \subset S$, it implies that $(\text{int } S) \cap S = \text{int } S$. (**)

Since $\text{int } \mathcal{C}S \subset \mathcal{C}S$, it implies that $(\text{int } \mathcal{C}S) \cap S = \phi$ (***)

~~So, $S = \text{int } S \cup$~~ Recall that $Bd(S) = \overline{S} \cap \overline{\mathcal{C}S}$. ①

Hence, $Bd(S) \cap S = \overline{S} \cap \overline{\mathcal{C}S} \cap S = S \cap \overline{\mathcal{C}S}$. So, $S = \text{int } S \cup (S \cap \overline{\mathcal{C}S})$

(*)

($\rightarrow$) Assume S is closed, So, $S = \overline{S}$

Let $p \in Bd(S)$. We will show that $p \in S$. ~~Since~~

Since $p \in \overline{S} \cap \overline{\mathcal{C}S}$ and $S = \overline{S}$, we can conclude that $p \in S$.

So, $Bd(S) \subset S$.

($\leftarrow$) Assume that $Bd\ S \subset S$. Therefore, $S = \text{int } S \cup Bd(S)$

by (*), (**), (***). Let $x \in \overline{S}$. If $x \notin S$ then $x \notin \text{int } S$.

Hence $x \in \text{int } \mathcal{C}S \cup Bd\ S$. Since $Bd\ S \subset S$, we can conclude

(by *)

that $x \in \text{int } \mathcal{C}S$. This is impossible by 16(c). So, $x \in S$.

Hence $\overline{S} = S$, and S is closed.

c) S is open $\longleftrightarrow$ S and its boundary are disjoint.

Proof

($\rightarrow$) Assume that S is open. By 15), we can conclude that $S = \text{int}\, S$.

By 17 a), we can conclude that $S \cap Bd(S) = \phi$.

($\leftarrow$) Assume that $S \cap (Bd\, S) = \phi$. By (*), (**), (***), we can conclude that $S = \text{int}\, S$. Since int S is open, we can conclude that S is open.