Chapter 2 Problems

1) Suppose that T is linear, such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$.

Determine $T\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right)$. Justify your answer.

2) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T(\vec{e_1}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(\vec{e_2}) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ and $T(\vec{e_3}) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. Justify your answer.

3) Use the definition of linear transformation to show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$ is a linear transformation.

4) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix}$.

a) What is the null space of T?

b) Is T one-to-one?

c) What is the range of T?

d) Is T onto?

5 Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and has property that $T(\vec{e_1}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T(\vec{e_2}) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

a) Determine whether T is one-to-one?

b) Determine whether T is onto?

6) Determine whether $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ is invertible. If so, find its inverse; if not, explain why not.

4 Find a matrix A, given the reduced row echelon form R of A and information about certain columns of A.

$$R = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ -2 \end{bmatrix}, \quad \vec{a}_4 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{a}_5 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

Solution

1) Step 1 Write $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Set $a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$.

Then we have $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$.

Since $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 2 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 3 \end{array}\right]$, we can conclude that $a = -3$ and $b = 3$.

Hence, $\begin{bmatrix} -3 \\ 3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Step 2 We will find $T\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right) = T\left((-3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = (-3)T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$
$$= (-3)\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ -9 \\ 6 \end{bmatrix}.$$

2) Recall that $T(\vec{x}) = A\vec{x}$ where A is the standard matrix of T.
Since $T(\vec{e}_1) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ and $T(\vec{e}_3) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, we can conclude that the standard matrix of T is $\begin{bmatrix} -2 & 0 & 2 \\ 1 & -3 & 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

3) i) Preserving vector addition.
Let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$.

Since $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = T\left(\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}\right) = \begin{bmatrix} (a+d)+(b+e) \\ c+f \end{bmatrix}$

and $T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + T\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = \begin{bmatrix} a+b \\ c \end{bmatrix} + \begin{bmatrix} d+e \\ f \end{bmatrix} = \begin{bmatrix} a+b+d+e \\ c+f \end{bmatrix}$,

it follows that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

ii) Preserving scalar multiplication

Let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and let α be a scalar.

$$T(\alpha\vec{u}) = T\left(\alpha\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}\right) = \begin{bmatrix} \alpha a + \alpha b \\ \alpha c \end{bmatrix} = \begin{bmatrix} \alpha(a+b) \\ \alpha c \end{bmatrix} = \alpha\begin{bmatrix} a+b \\ c \end{bmatrix}$$

$$= \alpha T(\vec{u}).$$

By i) & (ii), T is a linear transformation.

4) a) $\mathrm{Null}(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

b) Since $\mathrm{Null}(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, we can conclude that T is 1-1.

c) $\mathrm{Range}(T) = \mathrm{Span}\left\{ T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3) \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^3.$

d) Yes.

5) The standard matrix of T is $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$. Rank of such matrix is 1.

Hence T is not 1-1 and T is not onto.

6) Since $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$, it implies that

the given matrix is invertible and its inverse is $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$.

4. $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & -1 & 3 & -4 & -1 \\ 3 & 6 & 4 & -1 & -6 & -7 \\ -1 & -2 & -2 & 1 & 1 & 2 \end{bmatrix}$.

Note $\vec{a}_2 = 2\vec{a}_1$

$\vec{a}_4 = \vec{a}_1 - \vec{a}_3 \rightarrow \vec{a}_3 = \vec{a}_1 - \vec{a}_4$

$\vec{a}_6 = \vec{a}_1 - \vec{a}_3 + \vec{a}_5 \rightarrow \vec{a}_5 = \vec{a}_6 - \vec{a}_1 + \vec{a}_3$